FROBENIUS CHARACTER FORMULA AND SPIN GENERIC DEGREES FOR HECKE-CLIFFORD ALGEBRA

JINKUI WAN AND WEIQIANG WANG

ABSTRACT. The spin analogues of several classical concepts and results for Hecke algebras are established. A Frobenius type formula is obtained for irreducible characters of the Hecke-Clifford algebra. A precise characterization of the trace functions allows us to define the character table for the algebra. The algebra is endowed with a canonical symmetrizing trace form, with respect to which the spin generic degrees are formulated and shown to coincide with the spin fake degrees. We further provide a characterization of the trace functions and the symmetrizing trace form on the spin Hecke algebra which is Morita super-equivalent to the Hecke-Clifford algebra.

CONTENTS

1. INTRODUCTION

1.1. The Hecke algebras associated to symmetric groups or more general finite Weyl groups are symmetric algebras with canonical symmetrizing trace forms. The generic degrees defined in the framework of generic Hecke algebras have played an important role in finite groups of Lie type (first developed systematically by Lusztig [Lu]; also cf. Geck-Pfeiffer [GP2]). On the other hand, a Frobenius character formula for Hecke algebra associated to symmetric groups has been established in [Ram] (also see King-Wybourne [KW]), and the notion of character tables for Hecke algebras has been formulated by Geck and Pfeiffer [GP1].

The Hecke-Clifford algebra $\mathcal{H}^c_n$, which is a deformation of the algebra $\mathfrak{H}^c_n = \mathcal{C}_n \rtimes S_n$, admits a natural superalgebra structure, first appeared in Olshanski [Ol] who formulated a super queer version of the Schur-Jimbo duality. Its representation theory was subsequently developed by Jones-Nazarov [JN] for a generic quantum parameter v, and it is indeed closely related to the spin representations of the symmetric group developed

2010 *Mathematics Subject Classification.* Primary: 20C08, 20C25, 20C30.

by Schur [Sch]. In particular the irreducible characters ζ^λ of $\mathcal{H}^c_n$ (always understood in the $\mathbb{Z}_2$-graded sense in this paper) are parametrized by the strict partitions of n. This algebra $\mathcal{H}^c_n$ is also known to be Morita super-equivalent to a spin Hecke algebra $\mathcal{H}^-_n$ introduced by the second author [W], which is a deformation of Schur's spin symmetric group algebra.

1.2. Here is a quick summary of the main results of this paper. We first establish a Frobenius type formula for irreducible characters of the Hecke-Clifford algebra. We endow the Hecke-Clifford algebra with a canonical symmetrizing trace form $\daleth$, and then find an explicit shifted hook formula for the spin generic degrees for the Hecke-Clifford algebra by a novel and simple application of our Frobenius type character formula. The spin Hecke algebra is also shown to carry a natural symmetrizing trace form, which is compatible with $\daleth$ for Hecke-Clifford algebra via the Morita super-equivalence.

1.3. Let us describe in some detail. Our approach to obtaining a Frobenius type character formula for Hecke-Clifford algebra (Theorem 3.7) takes advantage of the Sergeev-Olshanski duality, and it is inspired by Ram's approach who obtained a Frobenius character formula for the type A Hecke algebra via the Schur-Jimbo duality. The symmetric functions arising in our Frobenius type formula are certain spin Hall-Littlewood functions introduced by the authors in [WW2], which are one-parameter deformation of Schur Q-functions. This should be compared to the appearance of Hall-Littlewood functions in [Ram].

We show that every trace function on the Hecke-Clifford algebra over the ring $\mathbf{A} = \mathbb{Z}[\frac{1}{2}][v, v^{-1}]$ is completely determined by its values on the standard elements parametrized by the odd partitions of n (see Theorem 4.7 and Corollary 4.9). This leads to well-defined notions of class polynomials and character table for Hecke-Clifford algebra, similar to those for Hecke algebras introduced by Geck-Pfeiffer [GP1].

The Hecke-Clifford algebra is a symmetric superalgebra endowed with a canonical symmetrizing trace form $\daleth$ (the choice of $\daleth$ is not obvious as it does not restrict to the well-known symmetrizing trace form on its type A Hecke subalgebra), and this allows us to formulate the *spin generic degrees* D^λ for the irreducible characters ζ^λ of $\mathcal{H}^c_n$. Recall the authors [WW1] formulated earlier a spin coinvariant algebra for the algebra $\mathfrak{H}^c_n$, and found a closed formula for the so-called *spin fake degrees* (this terminology appeared later in [WW3]). For a symmetric algebra $\mathcal{H}$ with a symmetrizing form, there exist elements called *Schur elements* (cf. [GP2, Theorem 7.2.1]) for irreducible characters, which can be used to determine when $\mathcal{H}$ is semsimple. These elements are closely related to the generic degrees in the case of usual Hecke algebras (cf. [GP2, Section 8.1.8]). The Schur elements for Hecke-Clifford algebra are computed explicitly (Theorem 5.8), and they do not lie in $\mathbf{A}$ in general. The spin generic degrees for Hecke-Clifford algebra are shown to be polynomials in the quantum parameter v and they match perfectly with the spin fake degrees (Theorem 5.10). This phenomenon is strikingly parallel to the classical result due to Steinberg [S] that the generic degrees for the type A Hecke algebras coincide with the fake degrees for symmetric groups (also cf. [Lu, GP2]).

We also succeed (see Theorem 6.6) in describing the space of trace functions of the spin Hecke algebra $\mathcal{H}^-_n$ introduced in [W]. The canonical trace form $\daleth^-$ on $\mathcal{H}^-_n$

corresponding to the form $\daleth$ on $\mathcal{H}_n^c$ under the Morita super-equivalence is characterized in a simple way (Theorem 6.10), in spite of the fact that the braid relation is deformed for $\mathcal{H}_n^-$ and the standard elements depend on the choices of reduced expressions of a given element.

1.4. Here is the layout of the paper. In Section 2, we review the Sergeev duality and Olshanski duality, and set up various notations needed in the remainder of the paper. In Section 3, using the Olshanski duality we compute a Frobenius type character formula for $\mathcal{H}_n^c$, whose specialization at $v = 1$ is equivalent to the classical character formula of Schur for spin symmetric groups. In Section 4, by a sequence of reductions we show that the trace functions are determined by their values on standard elements parametrized by odd partitions of n. In Section 5, the symmetrizing trace form $\daleth$ on Hecke-Clifford algebra is introduced, and the spin generic degrees are computed using the Frobenius character formula given in Section 3. Finally in Section 6, we describe the counterparts of Section 4 and part of Section 5 for spin Hecke algebras.

Acknowledgements. The first author was partially supported by NSFC-11101031, and she thanks Shun-Jen Cheng at Academia Sinica for support and providing an excellent atmosphere in the summer of 2011, where part of this paper was written. The second author was partially supported by NSF DMS-1101268.

2. The Sergeev-Olshanski duality

In this preliminary section, we shall introduce the Hecke-Clifford algebra, review the Sergeev-Olshanski duality, and set up notations to be used in later sections.

2.1. **Basics on superalgebras.** Let $\mathbb{F}$ be a field, which is always assumed to be of characteristic not equal to 2 in this paper. By a vector superspace over $\mathbb{F}$ we mean a $\mathbb{Z}_2$-graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $\dim V_{\bar{0}} = r$ and $\dim V_{\bar{1}} = m$, we write $\underline{\dim} V = r|m$. Given a homogeneous element $0 \neq v \in V$, we denote its degree by $|v| \in \mathbb{Z}_2$. An associative $\mathbb{F}$-superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ satisfies $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$. By an ideal I and respectively a module M of a superalgebra A, we always mean that I and M are $\mathbb{Z}_2$-graded, i.e., $I = (I \cap \mathcal{A}_{\bar{0}}) \oplus (I \cap \mathcal{A}_{\bar{1}})$, $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. The superalgebra $\mathcal{A}$ is called simple if it has no non-trivial ideals.

Let V be an $\mathbb{F}$-superspace with $\underline{\dim} V = r|m$, then

$$M(V) := \mathrm{End}_{\mathbb{F}}(V)$$

is a simple superalgebra. Assume now in addition $r = m$ and an odd automorphism J of V of order 2 is given. The subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$,

$$Q(V) = \{x \in \mathrm{End}_{\mathbb{F}}(V) \mid x \text{ and } J \text{ super-commute}\},$$

is also a simple superalgebra. Observe that the resulting superalgebras $Q(V)$ are isomorphic to each other for different automorphisms J.

An irreducible module V over an $\mathbb{F}$-superalgebra $\mathcal{A}$ is called *split irreducible* if $\mathbb{E} \otimes_{\mathbb{F}} V$ is irreducible over $\mathbb{E} \otimes_{\mathbb{F}} \mathcal{A}$ for any field extension $\mathbb{E} \supseteq \mathbb{F}$. A split irreducible $\mathcal{A}$-module V is of type $\mathtt{M}$ if $\mathrm{End}_{\mathbb{F}}(V)$ is one-dimensional and of type $\mathtt{Q}$ if $\mathrm{End}_{\mathbb{F}}(V)$ is two-dimensional. A superalgebra $\mathcal{A}$ is *split semisimple* if $\mathcal{A}$ is a direct sum of simple algebras of the form $M(V)$ and $Q(V)$ for various V.

Recall that given two superalgebras $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is naturally a superalgebra, with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b| \cdot |a'|}(aa') \otimes (bb') \qquad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}). \tag{2.1}$$

The following lemma can be found in [Jo] (where $\mathbb{F}$ is assumed to be an algebraically closed field).

Lemma 2.1. *Let V be a split irreducible $\mathcal{A}$-module and W be a split irreducible $\mathcal{B}$-module.*

(1) *If both V and W are of type $\mathtt{M}$, then $V \otimes W$ is a split irreducible $\mathcal{A} \otimes \mathcal{B}$-module of type $\mathtt{M}$.*
(2) *If one of V or W is of type $\mathtt{M}$ and the other is of type $\mathtt{Q}$, then $V \otimes W$ is a split irreducible $\mathcal{A} \otimes \mathcal{B}$-module of type $\mathtt{Q}$.*
(3) *If both V and W are of type $\mathtt{Q}$, then $V \otimes W$ is a sum of two isomorphic copies of a split irreducible module of type $\mathtt{M}$, which will be denoted by $2^{-1}V \otimes W$.*

Moreover, all split irreducible $\mathcal{A} \otimes \mathcal{B}$-modules arise as components of $V \otimes W$ for some choice of irreducibles V, W.

2.2. **The Sergeev duality.** In this subsection, we shall take $\mathbb{F} = \mathbb{C}$. Denote by $\mathcal{C}_n$ the Clifford superalgebra generated by odd elements $c_1, \ldots, c_n$ subject to the relations

$$c_i^2 = 1, c_i c_j = -c_j c_i, \quad 1 \le i \ne j \le n. \tag{2.2}$$

Denote by $\mathfrak{H}_n^c = \mathcal{C}_n \rtimes S_n$ the superalgebra generated by the even elements $s_1, \ldots, s_{n-1}$ and the odd elements $c_1, \ldots, c_n$ subject to (2.2), the standard Coxeter relation among $s_i = (i, i+1)$ for the symmetric group S_n, and the additional relations:

$$s_i c_i = c_{i+1} s_i, s_i c_j = c_j s_i, \quad 1 \le i, j \le n-1, j \ne i, i+1.$$

Denote by $\mathcal{P}_n$ the set of all partitions of n and by $\mathcal{CP}_n$ the set of compositions of n. Let $\mathcal{SP}_n$ (respectively, $\mathcal{OP}_n$) denote the set of strict (respectively, odd) partitions of n. For $\lambda \in \mathcal{P}_n$, denote by $\ell(\lambda)$ the length of λ and let

$$\delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even}, \\ 1, & \text{if } \ell(\lambda) \text{ is odd}. \end{cases}$$

Denote by Q_λ the Schur Q-functions associated to a strict partition λ (cf. [Mac, WW3]). It is known [Jo, Se] that there exists a characteristic map (cf. [WW3, (3.12)]) relating the representation theory of the algebra $\mathfrak{H}_n^c$ to the theory of symmetric functions, which can be viewed as a analog of the Frobenius characteristic map in the representation theory of symmetric groups. More precisely, for each strict partition λ of n, there exists an irreducible $\mathfrak{H}_n^c$-module U_1^λ which corresponds to the Schur Q-function Q_λ (up to some 2-power) under the characteristic map, and $\{U_1^\lambda \mid \lambda \in \mathcal{SP}_n\}$ forms a complete set of non-isomorphic irreducible $\mathfrak{H}_n^c$-modules. Furthermore, U_1^λ is of type $\mathtt{M}$ if $\delta(\lambda) = 0$ and is of type $\mathtt{Q}$ if $\delta(\lambda) = 1$. Denote by ζ_1^λ the character of U_1^λ for $\lambda \in \mathcal{SP}_n$.

The queer Lie superalgebra, denoted by $\mathfrak{q}(m)$, is the Lie superalgebra associated to the associative superalgebra $Q(m)$ with respect to the super-bracket. For convenience,

in the case when $\mathbb{F} = \mathbb{C}$, we shall take the odd involution

$$P = \sqrt{-1}\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \tag{2.3}$$

then $Q(m)$ and hence $\mathfrak{q}(m)$ will consist of $2m \times 2m$ matrices of the form:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{2.4}$$

where a and b are arbitrary $m \times m$ matrices over $\mathbb{C}$, and the rows and columns of (2.4) are labeled by the set

$$I(m|m) := \{-1, \ldots, -m, 1, \ldots, m\}.$$

Let $\mathfrak{g} = \mathfrak{q}(m)$. The even (respectively, odd) part $\mathfrak{g}_{\bar{0}}$ (respectively, $\mathfrak{g}_{\bar{1}}$) consists of those matrices of the form (2.4) with $b = 0$ (respectively, $a = 0$). Denote by E_{ij} for $i, j \in I(m|m)$ the standard elementary matrix with the (i,j)th entry being 1 and zero elsewhere. Fix the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ (respectively, $\mathfrak{n}^+$, $\mathfrak{n}^-$) is the subalgebra of $\mathfrak{g}$ which consists of matrices of the form (2.4) with a, b being arbitrary diagonal (respectively, upper triangular, lower triangular) matrices. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. We denote the standard basis for $\mathfrak{h}_{\bar{0}}$ by

$$H_i = E_{-i,-i} + E_{ii}, \qquad 1 \le i \le m.$$

Every finite-dimensional $\mathfrak{g}$-module is isomorphic to a highest weight module $V_1(\lambda)$ generated by a vector v_λ satisfying $\mathfrak{n}^+.v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}_{\bar{0}}$, for some $\lambda \in \mathfrak{h}_{\bar{0}}^*$. We have a weight space decomposition $V_1(\lambda) = \oplus_\mu V_1(\lambda)_\mu$, where a weight μ can be identified with an m-tuple $(\mu_1, \ldots, \mu_m)$. Let $x_1, \ldots, x_m$ be m independent variables. A character of a $\mathfrak{q}(m)$-module with weight space decomposition $M = \oplus M_\mu$ is defined to be

$$\mathrm{ch} M = \sum_{\mu=(\mu_1,\ldots,\mu_m)} \dim M_\mu x_1^{\mu_1} \cdots x_m^{\mu_m}.$$

We have a representation $(\omega_n, (\mathbb{C}^{m|m})^{\otimes n})$ of $\mathfrak{gl}(m|m)$, hence of its subalgebra $\mathfrak{q}(m)$, and we also have a representation $(\psi_n, (\mathbb{C}^{m|m})^{\otimes n})$ of the algebra $\mathfrak{H}_n^c$ defined by

$$\psi_n(s_i).(v_1 \otimes \ldots \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_n) = (-1)^{|v_i|\cdot|v_{i+1}|} v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_n,$$
$$\psi_n(c_i).(v_1 \otimes \ldots \otimes v_n) = (-1)^{(|v_1|+\ldots+|v_{i-1}|)} v_1 \otimes \ldots \otimes v_{i-1} \otimes P v_i \otimes \ldots \otimes v_n,$$

where $v_i, v_{i+1} \in \mathbb{C}^{m|m}$ are $\mathbb{Z}_2$-homogeneous. We recall a classical result of Sergeev.

Proposition 2.2. [Se, Theorem 3] *The algebras $\omega_n(U(\mathfrak{q}(m)))$ and $\psi_n(\mathfrak{H}_n^c)$ form mutual centralizers in $\mathrm{End}_{\mathbb{C}}((\mathbb{C}^{m|m})^{\otimes n})$. As an $U(\mathfrak{q}(m)) \otimes \mathfrak{H}_n^c$-module, we have*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \mathcal{SP}_n, \ell(\lambda) \le m} 2^{-\delta(\lambda)} V_1(\lambda) \otimes U_1^\lambda.$$

Moreover, the character of $V_1(\lambda)$ is given by

$$\mathrm{ch} V_1(\lambda) = 2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}} Q_\lambda(x_1, \ldots, x_m). \tag{2.5}$$

Associated to the formal variables $x_1, \ldots, x_m$, let D be the operator defined by

$$D = x_1^{H_1} \cdots x_m^{H_m}. \tag{2.6}$$

The operator D commutes with the action of $\mathfrak{H}_n^c$ on $(\mathbb{C}^{m|m})^{\otimes n}$, since the action of $\mathfrak{H}_n^c$ preserves the weight space decomposition. Proposition 2.2 has the following corollary.

Corollary 2.3. *For $h \in \mathfrak{H}_n^c$, the trace of the linear operator Dh on $(\mathbb{C}^{m|m})^{\otimes n}$ is*

$$tr(Dh) = \sum_{\lambda \in \mathcal{SP}_n, \ell(\lambda) \le m} 2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} Q_\lambda(x_1, \ldots, x_m) \zeta_1^\lambda(h).$$

2.3. **The Hecke-Clifford algebra $\mathcal{H}_n^c$.** Let v be an indeterminate. The Hecke-Clifford algebra $\mathcal{H}_n^c$ is the associative superalgebra over the field $\mathbb{C}(v^{\frac{1}{2}})$ with the even generators $T_1, \ldots, T_{n-1}$ and the odd generators $c_1, \ldots, c_n$ subject to the following relations:

$$\begin{aligned}
(T_i - v)(T_i + 1) = 0, &\quad 1 \le i \le n-1, \\
T_i T_j = T_j T_i, &\quad 1 \le i, j \le n-1, |i-j| > 1, \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, &\quad 1 \le i \le n-2, \\
c_i^2 = 1, c_i c_j = -c_j c_i, &\quad 1 \le i \ne j \le n, \\
T_i c_j = c_j T_i, &\quad j \ne i, i+1, 1 \le i \le n-1, 1 \le j \le n, \\
T_i c_i = c_{i+1} T_i, &\quad 1 \le i \le n-1.
\end{aligned}$$

Note that the specialization of $\mathcal{H}_n^c$ at $v = 1$ recovers $\mathfrak{H}_n^c$. Denote

$$[n] := \{1, \ldots, n\}.$$

For an (ordered) subset $I = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$, we denote $C_I = c_{i_1} c_{i_2} \ldots c_{i_k}$. By convention, $C_\emptyset = 1$. Then $\{C_I | \, I \subseteq [n]\}$ is a basis for the Clifford algebra $\mathcal{C}_n$. According to [JN, Proposition 2.1], the set $\{T_\sigma C_I \mid \sigma \in S_n, I \subseteq [n]\}$ forms a linear basis of the algebra $\mathcal{H}_n^c$.

Remark 2.4. Our definition of the Hecke-Clifford algebra $\mathcal{H}_n^c$ is slightly different from the algebra introduced in [Ol], where the quantum parameter q is used. The even generators $t_1, \ldots, t_{n-1}$ in [Ol] are related to $T_1, \ldots, T_{n-1}$ via $v = q^2$ and $t_i = v^{-\frac{1}{2}} T_i$, for $1 \le i \le n-1$.

Write $[k]_v = \dfrac{v^k - v^{-k}}{v - v^{-1}}$ for $k \in \mathbb{Z}_+$. Let

$$\mathbb{K} := \mathbb{C}(v^{\frac{1}{2}})\big(\sqrt{[1]_v}, \ldots, \sqrt{[n]_v}\big)$$

be the field extension of $\mathbb{C}(v^{\frac{1}{2}})$, and denote $\mathcal{H}_{n,\mathbb{K}}^c := \mathbb{K} \otimes_{\mathbb{C}(v^{\frac{1}{2}})} \mathcal{H}_n^c$.

Proposition 2.5. [JN, Corollary 6.8] *The $\mathbb{K}$-superalgebra $\mathcal{H}_{n,\mathbb{K}}^c$ is split semisimple. For each $\lambda \in \mathcal{SP}_n$, there exists an irreducible representation U^λ of $\mathcal{H}_{n,\mathbb{K}}^c$ with character ζ^λ such that $\{U^\lambda \mid \lambda \in \mathcal{SP}_n\}$ forms a complete set of nonisomorphic irreducible $\mathcal{H}_{n,\mathbb{K}}^c$-modules. Moreover, the specialization at $v = 1$ of ζ^λ gives ζ_1^λ for $\lambda \in \mathcal{SP}_n$.*

Remark 2.6. The precise form of the field $\mathbb{K}$ is not relevant in this paper and we could simply take $\mathbb{K}$ to be the algebraic closure of $\mathbb{C}(v^{\frac{1}{2}})$ as well. The construction of the irreducible $\mathcal{H}^c_{n,\mathbb{K}}$-modules U^λ in [JN] can be streamlined by a straightforward v-analogue of the semiformal form construction of the irreducible $\mathfrak{H}^c_n$-modules given in [HKS] and independently in [Wan] (cf. [WW3, Section 7]). One advantage of the latter approach is that it works equally well for the affine Hecke-Clifford algebra.

2.4. **The action of $\mathcal{H}^c_n$ on $V^{\otimes n}$.** Let $V=\mathbb{K}^{m|m}$ be the vector $\mathbb{K}$-superspace with $\underline{\dim}_{\mathbb{K}} V = m|m$. The standard basis $e_{-m},\dots,e_{-1},e_1,\dots,e_m$ of $\mathbb{C}^{m|m}$ can be naturally regarded as a $\mathbb{K}$-basis of V and the operator D is defined on $V^{\otimes n}$. Meanwhile $\{E_{ij} \mid i,j\in I(m|m)\}$ can be regarded as the standard basis of $\mathrm{End}_{\mathbb{K}}(V)$ with respect to the basis $\{e_{-m},\dots,e_{-1},e_1,\dots,e_m\}$ of V, i.e., $E_{ij}(e_k)=\delta_{jk}e_i$ for $k\in I(m|m)$. Following [Ol], we set

$$\begin{aligned}\Theta &= \sqrt{-1}\sum_{1\le a\le m}(E_{-a,a}-E_{a,-a}),\\ Q &= \sum_{i,j\in I(m|m)} \mathrm{sgn}(j)E_{ij}\otimes E_{ji},\\ S &= v^{\frac{1}{2}}\sum_{i\le j\in I(m|m)} S_{ij}\otimes E_{ij}\in \mathrm{End}_{\mathbb{K}}(V^{\otimes 2}),\end{aligned}\tag{2.7}$$

where S_{ij} are defined as follows:

$$S_{aa}=1+(v^{\frac{1}{2}}-1)(E_{aa}+E_{-a,-a}),\quad 1\le a\le m,\tag{2.8}$$

$$S_{-a,-a}=1+(v^{-\frac{1}{2}}-1)(E_{aa}+E_{-a,-a}),\quad 1\le a\le m,\tag{2.9}$$

$$S_{ab}=(v^{\frac{1}{2}}-v^{-\frac{1}{2}})(E_{ba}+E_{-b,-a}),\quad 1\le a<b\le m,\tag{2.10}$$

$$S_{-b,-a}=-(v^{\frac{1}{2}}-v^{-\frac{1}{2}})(E_{ab}+E_{-a,-b}),\quad 1\le a<b\le m,\tag{2.11}$$

$$S_{-b,a}=-(v^{\frac{1}{2}}-v^{-\frac{1}{2}})(E_{-a,b}+E_{a,-b}),\quad 1\le a,b\le m.\tag{2.12}$$

We remark that our definition of the operator S is a $v^{\frac{1}{2}}$ multiple of the original operator defined in [Ol] (see Remark 2.4, (2.13) and Proposition 2.7 below).

To endomorphisms $A\in\mathrm{End}_{\mathbb{K}}(V)$ and $C=\sum_\alpha A_\alpha\otimes B_\alpha\in\mathrm{End}_{\mathbb{K}}(V^{\otimes 2})$, we associate the following elements in $\mathrm{End}_{\mathbb{K}}(V^{\otimes n})$:

$$\begin{aligned}A^k &= \mathrm{I}^{\otimes k-1}\otimes A\otimes \mathrm{I}^{n-k},\\ C^{j,k} &= \sum_\alpha A^j_\alpha B^k_\alpha,\qquad 1\le j\ne k\le n.\end{aligned}$$

Olshanski [Ol] introduced the quantum deformation $U_v(\mathfrak{q}(m))$ of the universal enveloping algebra of $\mathfrak{q}(m)$, which is a $\mathbb{K}$-algebra with generators L_{ij} for $i,j\in I(m|m)$ with $i\le j$ subject to certain explicit relations (which we do not need here). Define $\Omega_n: U_v(\mathfrak{q}(m))\to \mathrm{End}_{\mathbb{K}}(V^{\otimes n})$ by letting

$$\Omega_n(L_{ij})=S_{ij},\qquad \text{for } i\le j\in I(m|m).\tag{2.13}$$

Then it is known [Ol] that Ω_n is an algebra homomorphism and hence defines a representation $(\Omega_n, V^{\otimes n})$ of $U_v(\mathfrak{q}(m))$, which is a deformation of the representation $(\omega_n, (\mathbb{C}^{m|m})^{\otimes n})$.

Proposition 2.7. [Ol, Theorems 5.2, 5.3] *Let $\bar{S} = QS \in \mathrm{End}_{\mathbb{K}}(V^{\otimes 2})$. Then there exists a representation $(\Psi_n, V^{\otimes n})$ of the Hecke-Clifford algebra $\mathcal{H}^c_{n,\mathbb{K}}$ defined by*

$$\Psi_n(T_j) = \bar{S}^{j,j+1}, \quad \Psi_n(c_k) = \Theta^k,$$

where $1 \le j \le n-1$ and $1 \le k \le n$. The algebras $\Omega_n(U_v(\mathfrak{q}(m)))$ and $\Psi_n(\mathcal{H}^c_{n,\mathbb{K}})$ form mutual centralizers in $\mathrm{End}_{\mathbb{K}}(V^{\otimes n})$. Moreover, as a $U_v(\mathfrak{q}(m)) \otimes \mathcal{H}^c_{n,\mathbb{K}}$-module, we have a multiplicity-free decomposition

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \mathcal{SP}_n, \ell(\lambda) \le m} 2^{-\delta(\lambda)} V(\lambda) \otimes U^\lambda,$$

where $V(\lambda)$'s are pairwise non-isomorphic irreducible $U_v(\mathfrak{q}(m))$-modules.

The operator D in (2.6) commutes with the action of $\mathcal{H}^c_{n,\mathbb{K}}$ on $V^{\otimes n}$, since the action of $\mathcal{H}^c_{n,\mathbb{K}}$ preserves the weight space decomposition. We have the following generalization of Corollary 2.3.

Proposition 2.8. *For $h \in \mathcal{H}^c_{n,\mathbb{K}}$, we have*

$$\mathrm{tr}(Dh) = \sum_{\lambda \in \mathcal{SP}_n, \ell(\lambda) \le m} 2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} Q_\lambda(x_1, \ldots, x_m) \zeta^\lambda(h).$$

Proof. Fix $\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{CP}_n$. Then the weight subspace $(V^{\otimes n})_\mu$ of $V^{\otimes n}$ has a basis given by $e_{i_1} \otimes \cdots \otimes e_{i_n}$ with $i_1, \ldots, i_n \in I(m|m)$ and $\sharp\{j||i_j| = k\} = \mu_k$ for $1 \le k \le m$. The operator D acts on $V^{\otimes n}_\mu$ as $(x_1^{\mu_1} \cdots x_m^{\mu_m}) \cdot \mathrm{I}$. On the other hand, $(V^{\otimes n})_\mu$ is stable under the action of $\mathcal{H}^c_{n,\mathbb{K}}$ and hence it can be decomposed as the direct sum of the irreducible module U^λ. This implies that the trace of Dh on $(V^{\otimes n})_\mu$ can be written as

$$\mathrm{tr} Dh|_{(V^{\otimes n})_\mu} = x_1^{\mu_1} \cdots x_m^{\mu_m} \sum_{\lambda \in \mathcal{SP}_n} f_{\lambda\mu} \zeta^\lambda(h)$$

for some $f_{\lambda\mu} \in \mathbb{Z}_+$. This holds for all $\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{CP}_n$ and hence

$$\mathrm{tr}(Dh) = \sum_{\lambda \in \mathcal{SP}_n} f_\lambda(x_1, \ldots, x_m) \zeta^\lambda(h),$$

where $f_\lambda(x_1, \ldots, x_m) = \sum_{\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{CP}_n} f_{\lambda\mu} x_1^{\mu_1} \ldots x_m^{\mu_m}$. Specializing $v = 1$ and using Proposition 2.5, we obtain that, for $h \in \mathfrak{H}^c_n$,

$$\mathrm{tr}(Dh) = \sum_{\lambda \in \mathcal{SP}_n} f_\lambda(x_1, \ldots, x_m) \zeta_1^\lambda(h).$$

Comparing with Corollary 2.3 and noting the linear independence of irreducible characters ζ_1^λ for $\lambda \in \mathcal{SP}_n$, one can deduce that $f_\lambda(x_1, \ldots, x_m) = Q_\lambda(x_1, \ldots, x_m)$ if $\ell(\lambda) \le m$ and $f_\lambda(x_1, \ldots, x_m) = 0$ otherwise. $\square$

3. A Frobenius type character formula

In this section, we shall formulate and establish a Frobenius type formula for the irreducible characters of the Hecke-Clifford algebra.

3.1. A formula for $\mathrm{tr}(DT_{w_{(n)}})$.

Recall that $\bar{S} = QS \in \mathrm{End}(V^{\otimes 2})$.

Lemma 3.1. *The following formula holds for $k, \ell \in I(m|m)$:*

$$\bar{S}(e_k \otimes e_\ell) =$$
$$\begin{cases} ve_\ell \otimes e_k + (v-1)e_{-k} \otimes e_{-\ell}, & \text{if } k = \ell \geq 1, \\ -e_\ell \otimes e_k, & \text{if } k = \ell \leq -1, \\ e_\ell \otimes e_k, & \text{if } k = -\ell \geq 1, \\ ve_\ell \otimes e_k + (v-1)e_k \otimes e_\ell, & \text{if } k = -\ell \leq -1, \\ v^{\frac{1}{2}} e_\ell \otimes e_k + (v-1)e_{-k} \otimes e_{-\ell} + (v-1)e_k \otimes e_\ell, & \text{if } |k| < |\ell| \text{ and } \ell \geq 1, \\ v^{\frac{1}{2}} \mathrm{sgn}(k) e_\ell \otimes e_k, & \text{if } |k| < |\ell| \text{ and } \ell \leq -1, \\ v^{\frac{1}{2}} e_\ell \otimes e_k + \mathrm{sgn}(\ell)(v-1)e_{-k} \otimes e_{-\ell}, & \text{if } |\ell| < |k| \text{ and } k \geq 1, \\ \mathrm{sgn}(\ell) e_\ell \otimes e_k + (v-1)e_k \otimes e_\ell, & \text{if } |\ell| < |k| \text{ and } k \leq -1. \end{cases}$$

Proof. By (2.7), we compute that

(3.1)

$$\begin{aligned} & v^{-\frac{1}{2}} S(e_k \otimes e_\ell) \\ &= \sum_{i \leq j \in I(m|m)} (S_{ij} \otimes E_{ij})(e_k \otimes e_\ell) \\ &= \sum_{i \leq j \in I(m|m)} (-1)^{|E_{ij}| \cdot |e_k|} S_{ij}(e_k) \otimes E_{ij}(e_\ell) \\ &= \sum_{i=-m}^{\ell} (-1)^{|E_{i\ell}| \cdot |e_k|} S_{i\ell}(e_k) \otimes e_i \\ &= \begin{cases} S_{\ell\ell}(e_k) \otimes e_\ell + \sum_{1 \leq i < \ell} S_{i\ell}(e_k) \otimes e_i + \sum_{-m \leq i \leq -1} (-1)^{|e_k|} S_{i\ell}(e_k) \otimes e_i, & \text{if } \ell \geq 1, \\ S_{\ell\ell}(e_k) \otimes e_\ell + \sum_{-m \leq i < \ell} S_{i\ell}(e_k) \otimes e_i, & \text{if } \ell \leq -1. \end{cases} \end{aligned}$$

Then the lemma is proved case-by-case using the definition of S_{ij} given in (2.8)-(2.12). Let us illustrate by checking in detail the case when $|k| < |\ell|, \ell \geq 1$. In this case, we have either $1 \leq k < \ell \leq m$ or $-\ell < k \leq -1$. If $1 \leq k < \ell \leq m$, then it follows by (2.10) and (2.12) that

$$v^{\frac{1}{2}} \sum_{1 \leq i < \ell} S_{i\ell}(e_k) \otimes e_i = (v-1)e_\ell \otimes e_k,$$
$$v^{\frac{1}{2}} \sum_{-m \leq i \leq -1} (-1)^{|e_k|} S_{i\ell}(e_k) \otimes e_i = -(v-1)e_{-\ell} \otimes e_{-k},$$

and hence by (3.1) we obtain that

$$S(e_k \otimes e_\ell) = v^{\frac{1}{2}} e_k \otimes e_\ell + (v-1)e_\ell \otimes e_k - (v-1)e_{-\ell} \otimes e_{-k}.$$

Therefore,

$$\begin{aligned}\bar{S}(e_k\otimes e_\ell)=&Q\big(v^{\frac12}e_k\otimes e_\ell+(v-1)e_\ell\otimes e_k-(v-1)e_{-\ell}\otimes e_{-k}\big)\\ =&v^{\frac12}e_\ell\otimes e_k+(v-1)e_k\otimes e_\ell+(v-1)e_{-k}\otimes e_{-\ell}.\end{aligned}$$

If $-\ell<k\le -1$, then by (2.10) and (2.12) we have

$$v^{\frac12}\sum_{1\le i<\ell}S_{i\ell}(e_k)\otimes e_i=(v-1)e_{-\ell}\otimes e_{-k},$$
$$v^{\frac12}\sum_{-m\le i\le -1}(-1)^{|e_k|}S_{i\ell}(e_k)\otimes e_i=(v-1)e_\ell\otimes e_k,$$

and hence by (3.1) we have

$$S(e_k\otimes e_\ell)=v^{\frac12}e_k\otimes e_\ell+(v-1)e_{-\ell}\otimes e_{-k}+(v-1)e_\ell\otimes e_k.$$

Therefore,

$$\bar{S}(e_k\otimes e_\ell)=v^{\frac12}e_\ell\otimes e_k+(v-1)e_{-k}\otimes e_{-\ell}+(v-1)e_k\otimes e_\ell.$$

The remaining cases can be verified similarly, and we skip the detail. □

For a composition $\gamma=(\gamma_1,\dots,\gamma_\ell)$ of n, let

$$\begin{aligned}&T_{\gamma,j}=T_{\gamma_1+\ldots+\gamma_{j-1}+1}T_{\gamma_1+\ldots+\gamma_{j-1}+2}\cdots T_{\gamma_1+\cdots+\gamma_j-1},\quad 1\le j\le\ell,\\ &T_{w_\gamma}=T_{\gamma,1}T_{\gamma,2}\cdots T_{\gamma,\ell}.\end{aligned}\tag{3.2}$$

Equivalently, T_{w_γ} is the Hecke algebra element corresponding to the permutation

$$w_\gamma=(1,\dots,\gamma_1)(\gamma_1+1,\dots,\gamma_1+\gamma_2)\cdots(\gamma_1+\cdots+\gamma_{\ell-1}+1,\dots,\gamma_1+\cdots+\gamma_\ell).\tag{3.3}$$

For $\underline{i}=(i_1,i_2,\dots,i_n)\in I(m|m)^n$ satisfying $i_1\le i_2\le\cdots\le i_n$, denote

$$\begin{aligned}f(\underline{i})&=\sharp\{1\le k\le n|i_k=i_{k+1}\ge 1\},\\ g(\underline{i})&=\sharp\{1\le k\le n|i_k=i_{k+1}\le -1\},\\ h(\underline{i})&=\sharp\{1\le k\le n|i_k<i_{k+1}\}.\end{aligned}$$

We denote by $T(u)|_u$ the coefficient of u in the linear expansion of $T(u)$ in terms of the basis $\{e_{i_1}\otimes\cdots\otimes e_{i_n}\mid i_1,\dots,i_n\in I(m|m)\}$ of $V^{\otimes n}$, for a linear operator $T\in\mathrm{End}_{\mathbb{K}}(V^{\otimes n})$ and a basis element u.

Lemma 3.2. *The trace of the operator $DT_{w_{(n)}}$ on $V^{\otimes n}$ is given by*

$$\mathrm{tr}(DT_{w_{(n)}})=\sum_{\underline{i}}v^{f(\underline{i})}(-1)^{g(\underline{i})}(v-1)^{h(\underline{i})}x_{|i_1|}\cdots x_{|i_n|},$$

where the summation is over the n-tuples $\underline{i}=(i_1,i_2,\dots,i_n)\in I(m|m)^n$ which satisfy $i_1\le i_2\le\cdots\le i_n$.

Proof. It suffices to show that, for $u=e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_n}$,

$$(DT_{w_{(n)}}u)|_u=\begin{cases}v^{f(\underline{i})}(-1)^{g(\underline{i})}v^{h(\underline{i})}x_{|i_1|}\cdots x_{|i_n|}, & \text{if } i_1\le i_2\le\cdots\le i_n,\\ 0, & \text{otherwise.}\end{cases}$$

By definition, we have $T_{w_{(n)}} = T_{w_{(n-1)}} T_{n-1}$. It follows by Proposition 2.7 and Lemma 3.1 that

$$
\begin{aligned}
&(DT_{w_{(n)}} u)|_u \\
=&x_{|i_1|} \cdots x_{|i_n|} \cdot (T_{w_{(n)}} u)|_u \\
=&x_{|i_1|} \cdots x_{|i_n|} \cdot T_{w_{(n-1)}} \big(e_{i_1} \otimes \cdots \otimes e_{i_{n-2}} \otimes \bar{S}(e_{i_{n-1}} \otimes e_{i_n})\big)|_u \\
=&\begin{cases}
v x_{|i_1|} \cdots x_{|i_n|} T_{w_{(n-1)}} (e_{i_1} \otimes \cdots \otimes e_{i_{n-1}})|_{e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}}, & \text{if } i_{n-1} = i_n \geq 1, \\
-x_{|i_1|} \cdots x_{|i_n|} T_{w_{(n-1)}} (e_{i_1} \otimes \cdots \otimes e_{i_{n-1}})|_{e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}}, & \text{if } i_{n-1} = i_n \leq -1, \\
(v-1) x_{|i_1|} \cdots x_{|i_n|} T_{w_{(n-1)}} (e_{i_1} \otimes \cdots \otimes e_{i_{n-1}})|_{e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}}, & \text{if } i_{n-1} < i_n, \\
0, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

Now the lemma follows by induction on n. □

For $s \geq 1$ and a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$, set

$$
\begin{aligned}
&\Delta_0 = 1, \\
&\Delta_s = v^{s-1} + (-1)^{s-1} + (v-1) \sum_{t=1}^{s-1} v^{t-1} (-1)^{s-t-1} = \frac{2(v^s - (-1)^s)}{v+1}, \\
&\Delta_\alpha = \Delta_{\alpha_1} \Delta_{\alpha_2} \cdots \Delta_{\alpha_\ell}.
\end{aligned}
$$

Let m_μ denote the monomial symmetric function associated to a partition μ.

Proposition 3.3. *The trace of the operator $DT_{w_{(n)}}$ on $V^{\otimes n}$ is given by*

$$\mathrm{tr}(DT_{w_{(n)}}) = \sum_{\mu \in \mathcal{P}_n} \Delta_\mu (v-1)^{\ell(\mu)-1} m_\mu(x_1, \dots, x_m).$$

Proof. Given $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{CP}_n$, denote by $\Gamma(\alpha)$ the set consisting of $\underline{i} = (i_1, \dots, i_n) \in I(m|m)^n$ satisfying $i_1 \leq \cdots \leq i_n$ and $\sharp\{j|\ |i_j| = k\} = \alpha_k$ for $1 \leq k \leq m$. By Lemma 3.2, the coefficient of the monomial $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ in $\mathrm{tr}(DT_{w_{(n)}})$, denoted by $\mathrm{tr}(DT_{w_{(n)}})|_{x^\alpha}$, is given by

$$\mathrm{tr}(DT_{w_{(n)}})|_{x^\alpha} = \sum_{\underline{i} \in \Gamma(\alpha)} v^{f(\underline{i})} (-1)^{g(\underline{i})} (v-1)^{h(\underline{i})}.$$

Now for $\underline{i} = (i_1, \dots, i_n)$, we let

$$\beta_k^+ = \{j | i_j = k\},\ \beta_k^- = \{j | i_j = -k\}, \qquad 1 \leq k \leq m.$$

Then each $\underline{i}$ in $\Gamma(\alpha)$ is uniquely determined by the pair $(\beta^+, \beta^-) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^m$, where $\beta^\pm = (\beta_1^\pm, \dots, \beta_m^\pm)$ satisfies $\beta_k^+ + \beta_k^- = \alpha_k$ for $1 \leq k \leq m$. In terms of these notations, we rewrite

$$
\begin{aligned}
v^{f(\underline{i})} &= \prod_{1 \leq k \leq m, \beta_k^+ \geq 1} v^{\beta_k^+ - 1}, \\
(-1)^{g(\underline{i})} &= \prod_{1 \leq k \leq m, \beta_k^- \geq 1} (-1)^{\beta_k^- - 1}, \\
(v-1)^{h(\underline{i})} &= (v-1)^{N(\beta^\pm) - 1},
\end{aligned}
$$

where we have denoted

$$N(\beta^{\pm}) = \sharp\{1 \le k \le m \mid \beta_k^+ > 0\} + \sharp\{1 \le k \le m \mid \beta_k^- > 0\}.$$

Let $\Omega(\alpha)$ denote the set which consists of all the pairs (β^+, β^-) satisfying $\beta_k^+ + \beta_k^- = \alpha_k$, for $1 \le k \le m$, and let $\ell(\alpha)$ denote the number of nonzero parts in α. Then

$$\begin{aligned}
&\mathrm{tr}(DT_{w_{(n)}})|_{x^\alpha}\\
&= \sum_{\underline{i} \in \Gamma(\alpha)} v^{f(\underline{i})}(-1)^{g(\underline{i})}(v-1)^{h(\underline{i})}\\
&= \sum_{(\beta^+,\beta^-)\in\Omega(\alpha)} \prod_{1\le k\le m, \beta_k^+\ge 1} v^{\beta_k^+ - 1} \cdot \prod_{1\le k\le m, \beta_k^-\ge 1} (-1)^{\beta_k^- -1} \cdot (v-1)^{N(\beta^\pm)-1}\\
&=(v-1)^{-1} \sum_{(\beta^+,\beta^-)\in\Omega(\alpha)} \prod_{k=1}^m v^{\beta_k^+ - 1 + \delta_{\beta_k^+,0}} \cdot (-1)^{\beta_k^- - 1 + \delta_{\beta_k^-,0}} \cdot (v-1)^{2-\delta_{\beta_k^+,0} - \delta_{\beta_k^-,0}}\\
&=(v-1)^{\ell(\alpha)-1} \prod_{1\le k\le m, \alpha_k>0} \Big(v^{\alpha_k-1} + (-1)^{\alpha_k-1} + \sum_{j=1}^{\alpha_k-1} v^{j-1}(-1)^{\alpha_k-j-1}(v-1)\Big)\\
&=(v-1)^{\ell(\alpha)-1}\Delta_\alpha.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{tr}(DT_{w_{(n)}}) &= \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{Z}_+^m, \sum_k \alpha_k = n} (v-1)^{\ell(\alpha)-1}\Delta_\alpha x_1^{\alpha_1}\cdots x_m^{\alpha_m}\\
&= \sum_{\mu\in\mathcal{P}_n} (v-1)^{\ell(\mu)-1}\Delta_\mu m_\mu(x_1,\ldots,x_m).
\end{aligned}$$

The proposition is proved. $\square$

For $n \ge 0$ and $x = (x_1, \ldots, x_m)$, we define $g_n(x; v)$ by the following generating function in a variable t:

$$\sum_{n\ge 0} g_n(x;v)t^n = \prod_i \frac{1-tx_i}{1+tx_i}\cdot\frac{1+vtx_i}{1-vtx_i}, \tag{3.4}$$

and then set

$$\widetilde{g}_n(x;v) = \frac{1}{v-1} g_n(x;v). \tag{3.5}$$

Proposition 3.3 can be reformulated as follows.

Proposition 3.4. *For $n \ge 1$, we have $tr(DT_{w_{(n)}}) = \widetilde{g}_n(x;v)$.*

Proof. By Proposition 3.3, we have

$$\begin{aligned}
&(v-1)^{-1}+\sum_{n\geq 1}\mathrm{tr}(DT_{w_{(n)}})t^n\\
&\quad=\sum_n t^n\sum_{\mu\in\mathcal{P}_n}(v-1)^{\ell(\mu)-1}\Delta_\mu m_\mu(x_1,\ldots,x_m)\\
&\quad=\frac{1}{v-1}\prod_i\Big(1+\sum_{s\geq 1}(v-1)\Delta_s x_i^s t^s\Big)\\
&\quad=\frac{1}{v-1}\prod_i\left(1+\sum_{s\geq 1}(v-1)\frac{2v^s x_i^s t^s-2(-1)^s x_i^s t^s}{v+1}\right)\\
&\quad=\frac{1}{v-1}\prod_i\Big(1+\frac{2(v-1)}{v+1}\Big(\frac{vx_it}{1-vx_it}-\frac{-x_it}{1+x_it}\Big)\Big)\\
&\quad=\frac{1}{v-1}\prod_i\frac{1-x_it}{1+x_it}\cdot\frac{1+vx_it}{1-vx_it}.
\end{aligned}$$

This implies (and is indeed equivalent to) the proposition by using (3.4) and (3.5). □

Remark 3.5. The symmetric function $g_n(x;v)$ also appears as a special case of the spin Hall-Littlewood functions (i.e., the one associated to the one-row partition (n)) introduced in [WW2].

3.2. A Frobenius formula for characters of $\mathcal{H}_n^c$.

Lemma 3.6. *Assume that $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_\ell)$ is a composition of n. Then*

$$\mathrm{tr}(DT_{w_\gamma})=\prod_{1\leq j\leq\ell}\mathrm{tr}(DT_{\gamma,j}).$$

Proof. Observe that

$$T_{w_\gamma}=\prod_{1\leq j\leq\ell}T_{\gamma,j},$$

and $T_{\gamma,j}$'s commute with each other. By Proposition 2.7, we note that $T_{\gamma,1}$ acts only on the first γ_1 factors, $T_{\gamma,2}$ acts only on the subsequent γ_2 factors and so on. The lemma follows. □

For a partition $\mu=(\mu_1,\ldots,\mu_\ell)$, we define

$$\widetilde{g}_\mu(x;v)=\widetilde{g}_{\mu_1}(x;v)\widetilde{g}_{\mu_2}(x;v)\cdots\widetilde{g}_{\mu_\ell}(x;v).$$

We are ready to establish a Frobenius type formula for the characters ζ^λ of the Hecke-Clifford algebra $\mathcal{H}_{n,\mathbb{K}}^c$.

Theorem 3.7. *The following holds for each partition μ of n:*

$$\widetilde{g}_\mu(x;v)=\sum_{\lambda\in\mathcal{SP}_n}2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}}Q_\lambda(x)\zeta^\lambda(T_{w_\mu}).\tag{3.6}$$

Proof. By Lemma 3.6 and Proposition 3.4, one deduces that

$$\begin{aligned}\mathrm{tr}(DT_{w_\mu}) =&\mathrm{tr}(DT_{\mu,1})\cdots\mathrm{tr}(DT_{\mu,\ell})\\ =&\widetilde{g}_{\mu_1}(x;v)\cdots\widetilde{g}_{\mu_\ell}(x;v)=\widetilde{g}_\mu(x;v).\end{aligned}$$

This together with Proposition 2.8 implies the theorem. □

Remark 3.8. Recall the specialization at $q=1$ of the Frobenius character formula for type A Hecke algebra established in [Ram] recovers the original formula of Frobenius [Fr] for the irreducible characters of symmetric groups. For $r\geq 1$, we have

$$\widetilde{g}_r(x;v)|_{v=1}=\begin{cases}2p_r(x), & \text{for } r \text{ odd}\\ 0, & \text{for } r \text{ even,}\end{cases}$$

where p_r denotes the rth power sum symmetric function. Hence, the Frobenius formula in Theorem 3.7 specializes when $v=1$ to a character formula for $\mathfrak{H}^c_n$, which is essentially equivalent to Schur's original character formula for the spin symmetric groups [Sch] (cf. [Jo] and [WW3]).

4. Trace functions on Hecke-Clifford algebra

In this section, we will exhibit an explicit basis for the space of trace functions on the Hecke-Clifford algebra.

4.1. **The trace functions.** Let R be a commutative ring in which 2 is invertible. For an R-superalgebra $\mathcal{H}$ which is a free R-module, a *trace function* on $\mathcal{H}$ is an R-linear map $\phi:\mathcal{H}\to R$ satisfying

$$\phi(hh')=\phi(h'h) \text{ for } h,h'\in\mathcal{H},\qquad \phi(h)=0 \text{ for } h\in\mathcal{H}_{\bar{1}}.$$

For $h,h'\in\mathcal{H}$, define their commutator by $[h,h']=hh'-h'h$, and let $[\mathcal{H},\mathcal{H}]$ be the R-submodule of $\mathcal{H}$ spanned by all commutators (not super-commutators!). Observe that a linear map $\phi:\mathcal{H}\to R$ with $\phi(\mathcal{H}_{\bar{1}})=0$ is a trace function if and only if $[\mathcal{H},\mathcal{H}]_{\bar{0}}\subseteq \mathrm{Ker}\phi$. Thus, the space of trace functions on $\mathcal{H}$ is canonically isomorphic to the dual space $\mathrm{Hom}_R((\mathcal{H}/[\mathcal{H},\mathcal{H}])_{\bar{0}},R)$ of $(\mathcal{H}/[\mathcal{H},\mathcal{H}])_{\bar{0}}=\mathcal{H}_{\bar{0}}/[\mathcal{H},\mathcal{H}]_{\bar{0}}$.

Let $\mathbf{A}:=\mathbb{Z}[\frac{1}{2}][v,v^{-1}]\subseteq\mathbb{C}(v)$, and denote by $\mathcal{H}^c_{n,\mathbf{A}}$ the $\mathbf{A}$-subalgebra of $\mathcal{H}^c_n$ generated by $T_1,\ldots,T_{n-1}$ and $c_1,\ldots c_n$. Note that $\mathcal{H}^c_{n,\mathbf{A}}$ is $\mathbf{A}$-free and $\mathcal{H}^c_n=\mathbb{C}(v^{\frac{1}{2}})\otimes_{\mathbf{A}}\mathcal{H}^c_{n,\mathbf{A}}$.

The main result of this subsection is Theorem 4.7, the proof of which requires a sequence of lemmas. Recall T_{w_γ} from (3.2).

Lemma 4.1. *For each $I\subseteq[n]$ and $\sigma\in S_n$, there exists an $\mathbf{A}$-linear combination of the form*

$$\Gamma_{I,\sigma}=\sum_{J\subseteq[n],\gamma\in\mathcal{CP}_n}a_{(I,\sigma),(J,\gamma)}T_{w_\gamma}C_J,$$

where $\ell(w_\gamma)\leq\ell(\sigma)$ and $a_{(I,\sigma),(J,\gamma)}\in\mathbf{A}$, such that

$$T_\sigma C_I\equiv\Gamma_{I,\sigma}\mod [\mathcal{H}^c_{n,\mathbf{A}},\mathcal{H}^c_{n,\mathbf{A}}].$$

Proof. Let i be the smallest integer such that $\sigma(i) > i+1$. We shall use a double induction involving an induction on $\sigma(i)$ and a reverse induction on i. Observe that if there does not exist such i, this can be regarded as the case $i = n$ and σ must be equal to w_γ defined in (3.3) for some composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$.

Let $j = \sigma(i) - 1$. Since $\sigma(\sigma^{-1}(j)) = j = \sigma(i) - 1 > i$, the choice of i implies that $\sigma^{-1}(j) > i$. This together with $\sigma^{-1}(j+1) = i$ implies that $\sigma^{-1}(j) > \sigma^{-1}(j+1)$ and hence $\ell(\sigma^{-1}s_j) < \ell(\sigma^{-1})$, or equivalently, $\ell(s_j\sigma) < \ell(\sigma)$. Then

$$T_\sigma = T_{s_j} T_{s_j\sigma}.$$

The following holds by the defining relation of $\mathcal{H}^c_n$:

$$C_I T_{s_j} = T_{s_j} C_{I'} + \sum_{J\subseteq[n]} a_J C_J \tag{4.1}$$

with $a_J \in \mathbf{A}$ and $I' \subseteq [n]$. We now consider two cases separately.

(i) First assume that $\ell(s_j\sigma s_j) > \ell(s_j\sigma)$. Then $T_{s_j\sigma s_j} = T_{s_j\sigma}T_{s_j}$, and hence

$$\begin{aligned}
T_\sigma C_I &= T_{s_j} T_{s_j\sigma} C_I \\
&\equiv T_{s_j\sigma} C_I T_{s_j} \quad \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}] \\
&= T_{s_j\sigma} T_{s_j} C_{I'} + \sum_J a_J T_{s_j\sigma} C_J \qquad \text{by (4.1)} \\
&= T_{s_j\sigma s_j} C_{I'} + \sum_J a_J T_{s_j\sigma} C_J.
\end{aligned}$$

(ii) Assume that $\ell(s_j\sigma s_j) < \ell(s_j\sigma)$. In this case, we have $T_{s_j\sigma} = T_{s_j\sigma s_j}T_{s_j}$, and thus $T_\sigma = T_{s_j}T_{s_j\sigma} = T_{s_j}T_{s_j\sigma s_j}T_{s_j}$. Hence,

$$\begin{aligned}
T_\sigma C_I &= T_{s_j} T_{s_j\sigma s_j} T_{s_j} C_I \\
&\equiv T_{s_j\sigma s_j} T_{s_j} C_I T_{s_j} \quad \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}].
\end{aligned}$$

Again by (4.1) we compute that

$$\begin{aligned}
T_{s_j\sigma s_j} T_{s_j} C_I T_{s_j} &= T_{s_j\sigma s_j} T_{s_j} T_{s_j} C_{I'} + \sum_J a_J T_{s_j\sigma s_j} T_{s_j} C_J \\
&= T_{s_j\sigma s_j} T_{s_j}^2 C_{I'} + \sum_J a_J T_{s_j\sigma s_j} T_{s_j} C_J \\
&= (v-1) T_{s_j\sigma s_j} T_{s_j} C_{I'} + v T_{s_j\sigma s_j} C_{I'} + \sum_{J\subseteq[n]} a_J T_{s_j\sigma s_j} T_{s_j} C_J \\
&= (v-1) T_{s_j\sigma} C_{I'} + v T_{s_j\sigma s_j} C_{I'} + \sum_{J\subseteq[n]} a_J T_{s_j\sigma} C_J.
\end{aligned}$$

Thus, if $\ell(s_j\sigma s_j) < \ell(s_j\sigma)$, we have

$$T_\sigma C_I \equiv (v-1) T_{s_j\sigma} C_{I'} + v T_{s_j\sigma s_j} C_{I'} + \sum_{J\subseteq[n]} a_J T_{s_j\sigma} C_J \quad \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}].$$

Observe that in each case, the resulted permutations $\sigma' := s_j\sigma$ and $\sigma'' := s_j\sigma s_j$ satisfy $\sigma'(k) = \sigma(k) = \sigma''(k)$ for $1 \le k \le i-1 < j-1$, since $\sigma(k) \le k+1 < j$ due to

the choice of i. In addition,

$$\sigma'(i) = s_j(\sigma(i)) = s_j(j+1) = j = \sigma(i) - 1,$$
$$\sigma''(i) = s_j(\sigma s_j(i)) = s_j(\sigma(i)) = s_j(j+1) = j = \sigma(i) - 1.$$

Note that $\ell(\sigma') = \ell(\sigma) - 1$ in both cases. Moreover, $\ell(\sigma'') = \ell(\sigma') + 1 = \ell(\sigma)$ in case (i), while $\ell(\sigma'') < \ell(\sigma') < \ell(\sigma)$ in case (ii). This completes the induction step, and hence the lemma is proved. □

Recall $T_{w_{(n)}} = T_1 T_2 \dots T_{n-1}$. Denote by

$$T_i' := T_i - v + 1 = vT_i^{-1}.$$

Lemma 4.2. *The following identities hold in $\mathcal{H}^c_{n,\mathbf{A}}$:*

$$T_{w_{(n)}} c_i = c_{i+1} T_{w_{(n)}} \qquad \text{for } 1 \le i \le n-1.$$
$$T_{w_{(n)}} c_n = c_1 T_1' T_2' \dots T_{n-1}' + (v-1) \Big(c_2 T_{w_{(1)}} T_2' \dots T_{n-1}' + c_3 T_{w_{(2)}} T_3' \dots T_{n-1}' + \dots + c_{n-1} T_{w_{(n-2)}} T_{n-1}' + c_n T_{w_{(n-1)}} \Big).$$

Proof. The first identity follows directly from the defining relations in $\mathcal{H}^c_{n,\mathbf{A}}$. Since $T_{n-1} c_n = c_{n-1} T_{n-1} - (v-1)(c_{n-1} - c_n)$, we obtain that

$$T_{w_{(n)}} c_n = T_{w_{(n-1)}} c_{n-1} T_{n-1}' + (v-1) c_n T_{w_{(n-1)}}.$$

Now the second identity follows from this identity by induction on n. □

Lemma 4.3. *For $I \subseteq [n]$ with $|I|$ even, the following holds:*

$$T_{w_{(n)}} C_I \equiv \pm T_{w_{(n)}} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}.$$

(The precise signs here and in the subsequent similar expressions shall not be needed.)

Proof. Set $I = \{i_1, \dots, i_k\}$. Since $k = |I|$ is even, we have $i_1 \le n-1$ and hence by Lemma 4.2

$$T_{w_{(n)}} C_I = T_{w_{(n)}} c_{i_1} \cdots c_{i_k} = c_{i_1+1} T_{w_{(n)}} c_{i_2} \cdots c_{i_k}.$$

Therefore,

$$\begin{aligned} T_{w_{(n)}} C_I &\equiv T_{w_{(n)}} c_{i_2} \cdots c_{i_k} c_{i_1+1} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}} \\ &= \begin{cases} (-1)^{k-1} T_{w_{(n)}} c_{i_1+1} c_{i_2} \cdots c_{i_k}, & \text{if } i_1 + 1 < i_2, \\ (-1)^{k-2} T_{w_{(n)}} c_{i_3} \cdots c_{i_k}, & \text{if } i_1 + 1 = i_2. \end{cases} \end{aligned}$$

In this way, we reduce $T_{w_{(n)}} C_I$ to a similar expression with smaller $|I|$ or with an increased i_1. By repeating the above procedure, the lemma is proved. □

Lemma 4.4. *Let $\gamma = (\gamma_1, \gamma_2)$ be a composition of n with $\gamma_1, \gamma_2 > 0$. Suppose $I_1 = \{i_1, \dots, i_a\} \subseteq \{1, \dots, \gamma_1\}, I_2 = \{j_1, \dots, j_b\} \subseteq \{\gamma_1 + 1, \dots, \gamma_1 + \gamma_2\}$ such that $a + b$ is even. Then*

$$T_{w_\gamma} C_{I_1} C_{I_2} \equiv \begin{cases} 0 & \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}, \quad \text{if } a \text{ and } b \text{ are odd}, \\ \pm T_{w_\gamma} & \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}, \quad \text{if } a \text{ and } b \text{ are even}. \end{cases}$$

Proof. Since $a+b$ is even, a and b have the same parity. Note that

$$T_{w_\gamma} = T_{\gamma,1}T_{\gamma,2} = T_{\gamma,2}T_{\gamma,1},$$
$$T_{\gamma,1}C_{I_2} = C_{I_2}T_{\gamma,1}, \qquad T_{\gamma,2}C_{I_1} = C_{I_1}T_{\gamma,2}.$$

If both a and b are odd, then $C_{I_1}C_{I_2} = -C_{I_2}C_{I_1}$. It follows from these identities above that

$$\begin{aligned} T_{w_\gamma}C_{I_1}C_{I_2} &= T_{\gamma,1}C_{I_1}T_{\gamma,2}C_{I_2} \\ &\equiv T_{\gamma,2}C_{I_2}T_{\gamma,1}C_{I_1} = -T_{w_\gamma}C_{I_1}C_{I_2}, \end{aligned}$$

where the notation $\equiv$ here and in similar expressions below is always understood mod $[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}$. Therefore, $T_{w_\gamma}C_{I_1}C_{I_2} \equiv 0 \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}$.

If both a and b are even, then $1 \le i_1 \le \gamma_1 - 1$ and $\gamma_1 + 1 \le j_1 \le \gamma_1 + \gamma_2 - 1 = n-1$. By a similar analysis as in Lemma 4.3, one can obtain that

$$\begin{aligned} T_{w_\gamma}C_{I_1}C_{I_2} &= T_{\gamma,1}T_{\gamma,2}c_{i_1}c_{i_2}\cdots c_{i_a}c_{j_1}c_{j_2}\cdots c_{j_b} \\ &= -T_{\gamma,1}c_{i_1}T_{\gamma,2}c_{j_1}c_{i_2}\cdots c_{i_a}\cdots c_{j_2}c_{j_b} \\ &= -c_{i_1+1}c_{j_1+1}T_{\gamma,1}T_{\gamma,2}c_{i_2}\cdots c_{i_a}\cdots c_{j_2}c_{j_b}. \end{aligned}$$

Hence,

$$\begin{aligned} T_{w_\gamma}C_{I_1}C_{I_2} &= -c_{i_1+1}c_{j_1+1}T_{\gamma,1}T_{\gamma,2}c_{i_2}\cdots c_{i_a}c_{j_2}\cdots c_{j_b} \\ &\equiv -T_{\gamma,1}T_{\gamma,2}c_{i_2}\cdots c_{i_a}c_{j_2}\cdots c_{j_b}c_{i_1+1}c_{j_1+1} \\ &= \begin{cases} T_{\gamma,1}T_{\gamma,2}c_{i_1+1}c_{i_2}\cdots c_{i_a}c_{j_1+1}c_{j_2}\cdots c_{j_b}, & \text{if } i_1+1<i_2, j_1+1<j_2, \\ -T_{\gamma,1}T_{\gamma,2}c_{i_1+1}c_{i_2}\cdots c_{i_a}c_{j_3}\cdots c_{j_b}, & \text{if } i_1+1<i_2, j_1+1=j_2, \\ -T_{\gamma,1}T_{\gamma,2}c_{i_3}\cdots c_{i_a}c_{j_1+1}c_{j_2}\cdots c_{j_b}, & \text{if } i_1+1=i_2, j_1+1<j_2, \\ T_{\gamma,1}T_{\gamma,2}c_{i_3}\cdots c_{i_a}c_{j_3}\cdots c_{j_b}, & \text{if } i_1+1=i_2, j_1+1=j_2. \end{cases} \end{aligned}$$

In this way, we reduce $T_{w_\gamma}C_{I_1}C_{I_2}$ to a similar expression with smaller $|I_1|+|I_2|$ or with increased i_1 and j_1. Repeating the procedure, the proposition is proved. ☐

Given integers $a<b$, we shall denote by $[a..b]$ the set of integers k such that $a \le k \le b$. The following is a generalization of Lemmas 4.3 and 4.4.

Lemma 4.5. *Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$ be a composition of n, and let $I \subseteq [n]$ with $|I|$ even. Let $I_k = I \cap [(\gamma_1 + \ldots + \gamma_{k-1} + 1)..(\gamma_1 + \ldots + \gamma_k)]$ for $1 \le k \le \ell$. Then*

$$T_{w_\gamma}C_I \equiv \begin{cases} \pm T_{w_\gamma} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}, & \textit{if every } |I_k| \textit{ is even for } 1 \le k \le \ell, \\ 0 \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}, & \textit{otherwise.} \end{cases}$$

Proof. The lemma for $\ell = 1$ reduces to Lemma 4.3. So assume $\ell \ge 2$. If every $|I_k|$ is even for $1 \le k \le \ell$, then the lemma follows by a similar proof as in Lemma 4.4 (which is the special case when $\ell = 2$). Otherwise, suppose there exists $1 \le a \le \ell$ such that $|I_a|$ is odd. Without loss of generality and for the sake of simplifying notations, we assume $a = 1$. Let $b > 1$ be the smallest integer such that I_b is odd (such b exists since

$|I| = k$ is even). Therefore,

$$\begin{aligned} T_{w_\gamma} C_I &= (T_{\gamma,1} C_{I_1} T_{\gamma,2} C_{I_2} \cdots T_{\gamma,b-1} C_{I_{b-1}})(T_{\gamma,b} C_{I_b})(T_{\gamma,b+1} C_{I_{b+1}} \cdots T_{\gamma,\ell} C_{I_\ell}) \\ &= (T_{\gamma,1} C_{I_1})(T_{\gamma,b} C_{I_b})(T_{\gamma,2} C_{I_2} \cdots T_{\gamma,b-1} C_{I_{b-1}})(T_{\gamma,b+1} C_{I_{b+1}} \cdots T_{\gamma,\ell} C_{I_\ell}) \\ &= -(T_{\gamma,b} C_{I_b})(T_{\gamma,1} C_{I_1})(T_{\gamma,2} C_{I_2} \cdots T_{\gamma,b-1} C_{I_{b-1}})(T_{\gamma,b+1} C_{I_{b+1}} \cdots T_{\gamma,\ell} C_{I_\ell}) \\ &\equiv -(T_{\gamma,1} C_{I_1})(T_{\gamma,2} C_{I_2} \cdots T_{\gamma,b-1} C_{I_{b-1}})(T_{\gamma,b+1} C_{I_{b+1}} \cdots T_{\gamma,\ell} C_{I_\ell})(T_{\gamma,b} C_{I_b}) \\ &= -T_{w_\gamma} C_I. \end{aligned}$$

Hence, $T_{w_\gamma} C_I \equiv 0 \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]$. This completes the proof of the lemma. □

Denote by $\mathcal{H}_{n,\mathbf{A}}$ the subalgebra of $\mathcal{H}^c_{n,\mathbf{A}}$ generated by $T_1, \dots, T_{n-1}$, which is the Hecke algebra over $\mathbf{A}$ associated to the symmetric group S_n.

Lemma 4.6. *Suppose $\gamma = (\gamma_1, \dots, \gamma_\ell)$ is a composition of n and let $\mu = (\mu_1, \dots, \mu_\ell)$ be the partition obtained by a rearrangement of the parts of γ. Then*

$$T_{w_\gamma} \equiv T_{w_\mu} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}.$$

Proof. It is known (see [Ram, Theorem 5.1] or [GP2, Section 8.2]) that $T_{w_\gamma} \equiv T_{w_\mu} \mod [\mathcal{H}_{n,\mathbf{A}}, \mathcal{H}_{n,\mathbf{A}}]$. The lemma now follows since $[\mathcal{H}_{n,\mathbf{A}}, \mathcal{H}_{n,\mathbf{A}}] \subseteq [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}$. □

Recall that $\mathcal{OP}_n$ denotes the set of odd partitions of n.

Theorem 4.7. *For each $\sigma \in S_n$ and $I \subseteq [n]$ with $|I|$ even, there exist $f_{\sigma,I;\nu} \in \mathbf{A}$ such that*

$$T_\sigma C_I \equiv \sum_{\nu \in \mathcal{OP}_n} f_{\sigma,I;\nu} T_{w_\nu} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}. \tag{4.2}$$

Proof. By Lemma 4.1, Lemma 4.5 and Lemma 4.6, it suffices to show that, for each partition μ of n, there exists $f_{\mu;\nu} \in \mathbf{A}$ such that

$$T_{w_\mu} \equiv \sum_{\nu \in \mathcal{OP}_n} f_{\mu;\nu} T_{w_\nu} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}. \tag{4.3}$$

Let us assume for a moment that n is even. Set $y_n := c_1 c_2 \dots c_n$. By Lemma 4.2, we calculate that

$$y_n^{-1} T_{w_{(n)}} y_n = (c_1 c_2 \dots c_n)^{-1} (c_2 c_3 \dots c_n) T_{w_{(n)}} c_n = -c_1 T_{w_{(n)}} c_n.$$

By the second identity in Lemma 4.2 (and expanding the T_i' therein as a sum of monomials), $-c_1 T_{w_{(n)}} c_n$ can be written as $-T_{w_{(n)}} +$ a linear combination of elements of the form $c_k c_m T_\sigma$ with $\ell(\sigma) \le n-2 = \ell(w_{(n)}) - 1$. Equivalently, $y_n^{-1} T_{w_{(n)}} y_n = -c_1 T_{w_{(n)}} c_n$ can be written as $-T_{w_{(n)}} +$ a linear combination of elements of the form $T_\sigma c_i c_j$ with $\ell(\sigma) < \ell(w_{(n)})$.

Now we come to the proof of (4.3). Let us assume that μ is a partition of n with an even part μ_a for some $a \in \{1, \dots, \ell(\mu)\}$. Set

$$y := c_{(\mu_1 + \dots + \mu_{a-1} + 1)} c_{(\mu_1 + \dots + \mu_{a-1} + 2)} \cdots c_{(\mu_1 + \dots + \mu_{a-1} + \mu_a)}.$$

Then, the computation in the previous paragraph is applicable to $y^{-1} T_{\mu,a} y$, which in turn implies that

$$y^{-1} T_{w_\mu} y = -T_{w_\mu} + Z, \tag{4.4}$$

where Z is a linear combination of elements of the form $T_\sigma c_i c_j$ where $\ell(\sigma) < \ell(w_\mu)$. By Lemma 4.1, $T_\sigma c_i c_j$ is a linear combination of $T_{w_\lambda} C_J$ with $\ell(w_\lambda) \le \ell(\sigma)$. Hence, Z is a linear combination of elements of the form $T_{w_\lambda} C_I$ with $\ell(w_\lambda) < \ell(w_\mu)$. By Lemma 4.5,

$$\frac{1}{2} Z \equiv \text{ a linear combination of } T_{w_\lambda} \text{ with } \ell(w_\lambda) < \ell(w_\mu), \quad \mathrm{mod}\ [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}.$$

On the other hand, we have $T_{w_\mu} = (T_{w_\mu} y) y^{-1} \equiv y^{-1} T_{w_\mu} y \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}$. This together with (4.4) implies that

$$T_{w_\mu} \equiv \frac{1}{2} Z \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}.$$

Now the proof is completed by induction on the length $\ell(w_\mu)$. □

4.2. **The space of trace functions and character table of $\mathcal{H}^c_{n,\mathbf{A}}$.** Theorem 4.7 has the following implication.

Theorem 4.8. *$(\mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}])_{\bar{0}}$ is a free $\mathbf{A}$-module, with a basis consisting of the images of T_{w_ν} for $\nu \in \mathcal{OP}_n$ under the projection $\mathcal{H}^c_{n,\mathbf{A}} \to \mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]$.*

Proof. By Theorem 4.7, $(\mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}])_{\bar{0}}$ is spanned by the images of the elements T_{w_ν} with $\nu \in \mathcal{OP}_n$ under the projection $\mathcal{H}^c_{n,\mathbf{A}} \to \mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]$. Passing to the splitting field $\mathbb{K}$ for Hecke-Clifford algebra, the images of the elements T_{w_ν} with $\nu \in \mathcal{OP}_n$ remain to be a spanning set for $(\mathcal{H}^c_{n,\mathbb{K}}/[\mathcal{H}^c_{n,\mathbb{K}}, \mathcal{H}^c_{n,\mathbb{K}}])_{\bar{0}}$. By Proposition 2.5, $\mathcal{H}^c_{n,\mathbb{K}}$ is semisimple and its non-isomorphic irreducible characters are parametrized by $\mathcal{SP}_n$. It follows that the dimension of the space of trace functions on $\mathcal{H}^c_{n,\mathbb{K}}$ is

$$\dim_{\mathbb{K}} (\mathcal{H}^c_{n,\mathbb{K}}/[\mathcal{H}^c_{n,\mathbb{K}}, \mathcal{H}^c_{n,\mathbb{K}}])_{\bar{0}} = |\mathcal{SP}_n| = |\mathcal{OP}_n|.$$

Hence the images of T_{w_ν} with $\nu \in \mathcal{OP}_n$ are linearly independent in $(\mathcal{H}^c_{n,\mathbb{K}}/[\mathcal{H}^c_{n,\mathbb{K}}, \mathcal{H}^c_{n,\mathbb{K}}])_{\bar{0}}$ as well as in $(\mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}])_{\bar{0}}$. This proves the theorem. □

Corollary 4.9. *Every trace function $\phi : \mathcal{H}^c_{n,\mathbf{A}} \to \mathbf{A}$ is uniquely determined by its values on the elements T_{w_ν} for $\nu \in \mathcal{OP}_n$. Moreover, the polynomials $f_{\sigma,I;\nu}$ in (4.2) are uniquely determined by σ, I and ν.*

For $\sigma \in S_n$ and $I \subseteq [n]$ with $|I|$ even and $\nu \in \mathcal{OP}_n$, $f_{\sigma,I;\nu}$ are called *class polynomials*, and they are spin analogues of the class polynomials introduced by Geck-Pfeiffer [GP1, Definition 1.2(2)] (cf. [GP2, Section 8.2]) for Hecke algebras associated to finite Weyl groups. The square matrix

$$\left[\zeta^\lambda(T_{w_\nu})\right]_{\lambda \in \mathcal{SP}_n, \nu \in \mathcal{OP}_n}$$

is called the *character table* of the Hecke-Clifford algebra $\mathcal{H}^c_{n,\mathbb{K}}$. By Corollary 4.9 and the linear independence of irreducible characters ζ^λ for $\lambda \in \mathcal{SP}_n$, the square matrix $[\zeta^\lambda(T_{w_\nu})]_{\lambda,\nu}$ is invertible in $\mathbb{K}$.

Remark 4.10. Note that w_ν is a minimal length representative in the conjugacy class C in S_n of cycle type $\nu \in \mathcal{OP}_n$. Let w_C be another minimal length representative in the same conjugacy class. It is known from [GP1, Theorem 1.1] that $T_{w_\nu} \equiv T_{w_C} \mod [\mathcal{H}_{n,\mathbf{A}}, \mathcal{H}_{n,\mathbf{A}}]$ and hence $T_{w_\nu} \equiv T_{w_C} \mod [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]$ thanks to $[\mathcal{H}_{n,\mathbf{A}}, \mathcal{H}_{n,\mathbf{A}}] \subseteq [\mathcal{H}^c_{n,\mathbf{A}}, \mathcal{H}^c_{n,\mathbf{A}}]_{\bar{0}}$. Thus our definition of the character table of $\mathcal{H}^c_{n,\mathbb{K}}$ is independent of the

choice of minimal length representatives in conjugacy classes of cycle type being odd partitions of n. Moreover, specializing $v = 1$, the matrix $(\zeta^\lambda(T_{w_\nu}))_{\lambda\in\mathcal{SP}_n,\nu\in\mathcal{OP}_n}$ reduces to the character table of the algebra $\mathfrak{H}_n^c$ (cf. [WW3]).

For $\nu \in \mathcal{OP}_n$, define an $\mathbf{A}$-linear map $f_\nu : \mathcal{H}^c_{n,\mathbf{A}} \to \mathbf{A}$ by

$$f_\nu(T_\sigma C_I) = \begin{cases} f_{\sigma,I;\nu}, & \text{if } |I| \text{ is even} \\ 0, & \text{if } |I| \text{ is odd.} \end{cases}$$

Proposition 4.11. *For each $\nu \in \mathcal{OP}_n$, f_ν is a trace function on $\mathcal{H}^c_{n,\mathbf{A}}$ which satisfies*

$$f_\nu(T_{w_\rho}) = \delta_{\nu,\rho}, \qquad \text{for } \rho \in \mathcal{OP}_n. \tag{4.5}$$

Moreover, $\{f_\nu|\nu \in \mathcal{OP}_n\}$ is a basis for the space of trace functions on $\mathcal{H}^c_{n,\mathbf{A}}$.

Proof. Recall that the distinct irreducible characters of $\mathcal{H}^c_{n,\mathbb{K}} = \mathbb{K}\otimes_{\mathbf{A}}\mathcal{H}^c_{n,\mathbf{A}}$ are given by ζ^λ for $\lambda \in \mathcal{SP}_n$. By Theorem 4.7, for any $\lambda \in \mathcal{SP}_n, \sigma \in S_n$ and $I \subseteq [n]$ with $|I|$ even we have

$$\zeta^\lambda(T_\sigma C_I) = \sum_{\nu\in\mathcal{OP}_n} f_{\sigma,I;\nu}\zeta^\lambda(T_{w_\nu}) = \sum_{\nu\in\mathcal{OP}_n} f_\nu(T_\sigma C_I)\zeta^\lambda(T_{w_\nu}).$$

Then by the invertibility of the character table $(\zeta^\lambda(T_{w_\nu}))_{\lambda\in\mathcal{SP}_n,\nu\in\mathcal{OP}_n}$ we can write

$$f_\nu(T_\sigma C_I) = \sum_{\lambda\in\mathcal{SP}_n} g_{\lambda;\nu}\zeta^\lambda(T_\sigma C_I)$$

for some $g_{\lambda;\nu} \in \mathbb{K}$. Therefore f_ν is a trace function on $\mathcal{H}^c_{n,\mathbb{K}}$ and hence a trace function on $\mathcal{H}^c_{n,\mathbf{A}}$. Now (4.5) follows from the definition of f_ν and (4.2). Then by Theorem 4.8, $\{f_\nu|\nu \in \mathcal{OP}_n\}$ forms a basis of the space of trace functions on $\mathcal{H}^c_{n,\mathbf{A}}$. □

5. Spin generic degrees for Hecke-Clifford algebra

In this section, we shall introduce the spin generic degrees for the Hecke-Clifford algebra and show that it coincides with spin fake degrees associated to the spin symmetric groups introduced in [WW1, WW3].

5.1. **Basics on symmetric superalgebras.** Let $\mathcal{H}$ be an R-superalgebra which is free and of finite rank over a commutative ring R containing $\frac{1}{2}$. A trace function $\phi : \mathcal{H} \to R$ is called a *symmetrizing trace form* if the bilinear form

$$\mathcal{H}\times\mathcal{H} \longrightarrow R, \quad (h, h') \mapsto \phi(hh')$$

is non-degenerate, i.e., there exists a homogeneous basis $\mathcal{B}$ of $\mathcal{H}$ such that the determinant of matrix $(\phi(b_1b_2))_{b_1,b_2\in\mathcal{B}}$ is a unit in R. In this case, $(\mathcal{H}, \phi)$ or $\mathcal{H}$ is called a *symmetric superalgebra*.

Remark 5.1. Let $\mathcal{H} = M(V)$ or $\mathcal{H} = Q(V)$ over a field $\mathbb{F}$. Then every trace function on $\mathcal{H}$ is a scalar multiple of the usual matrix trace tr. Note that $(\mathcal{H}, \mathrm{tr})$ is symmetric.

In the remainder of this subsection, we assume that $\mathcal{H}$ is symmetric with a symmetrizing trace form ϕ, and describe some basic results for $\mathcal{H}$. Though most are straightforward superalgebra generalizations of the well-known classical results (cf. [GP2, Chapter 7]), we need to make precise a possible factor of 2 due to type $\mathtt{Q}$ simple $\mathcal{H}$-modules.

If $\mathcal{B}$ is a $\mathbb{Z}_2$-homogeneous basis for $\mathcal{H}$, we denote by $\mathcal{B}^\vee = \{b^\vee | b \in \mathcal{B}\}$ the dual basis, which is also homogenous and satisfies that $\phi(b^\vee b') = \delta_{b,b'}$. Suppose V, V' are $\mathcal{H}$-modules. For any homogenous map $f \in \mathrm{Hom}_R(V, V')$, we define $I(f) \in \mathrm{Hom}_R(V, V')$ by letting

$$I(f)(v) = \sum_{b \in \mathcal{B}} (-1)^{|f||b|} b^\vee f(bv), \qquad \text{for } v \in V.$$

It follows by essentially the same proof as for [GP2, Lemma 7.1.10] with appropriate superalgebra signs inserted that $I(f)$ is independent of the choice of the homogeneous basis $\mathcal{B}$, and moreover $I(f) \in \mathrm{Hom}_{\mathcal{H}}(V, V')$.

Let $\mathbb{F}$ be a filed of characteristic not equal to 2. From now on, we assume that $\mathcal{H}$ is a finite dimensional superalgebra over a field $\mathbb{F}$ with a symmetrizing trace ϕ. The following lemma is the superalgebra analogue of [GP2, Theorem 7.2.1], which can be proved in the same way.

Lemma 5.2. *Let V be a split irreducible $\mathcal{H}$-module. Then there exists a unique element $c_V \in \mathbb{F}$ such that*

$$I(f) = c_V \mathrm{tr}(f)\ \mathrm{id}_V, \qquad \text{for } f \in \mathrm{End}_{\mathbb{F}}(V)_{\bar{0}}.$$

The element c_V is called the *Schur element* of V. Let us compute the Schur element of the unique irreducible representation of the simple superalgebras over $\mathbb{F}$.

Example 5.3. *(1) Let $\mathcal{H} = Q(V)$ with $V = \mathbb{F}^{m|m}$. Clearly V is an irreducible $\mathcal{H}$-module of type $\mathtt{Q}$. Let $v_1, \ldots, v_m$ be a basis of $V_{\bar{0}}$ and $v_{-1}, \ldots, v_{-m}$ be a basis of $V_{\bar{1}}$, and let $J \in \mathrm{End}_{\mathbb{F}}(V)$ be the automorphism sending v_k to v_{-k} for $1 \le k \le m$. Then $\mathcal{H}$ consists of $2m \times 2m$ matrices of the form:*

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where a and b are arbitrary $m \times m$ matrices. Observe that $\mathcal{B} = \{g_{ij} := E_{i,j} + E_{-i,-j} | 1 \le i, j \le m\} \cup \{h_{ij} := E_{-i,j} - E_{i,-j} | 1 \le i, j \le m\}$ is a basis of $\mathcal{H}$ and the dual basis with respect to the usual matrix trace tr *is $\mathcal{B}^\vee = \{g_{ij}^\vee = \frac{1}{2} g_{ji} | 1 \le i, j \le m\} \cup \{h_{ij}^\vee = -\frac{1}{2} h_{ji} | 1 \le i, j \le m\}$. Then a direct computation shows that, for $f \in \mathrm{End}_{\mathbb{F}}(V)_{\bar{0}}$,*

$$I(f)(v_k) = \frac{\mathrm{tr}(f)}{2} v_k, \qquad \text{for } k \in I(m|m).$$

By Lemma 5.2, the Schur element of the irreducible $\mathcal{H}$-module V (with respect to the usual matrix trace) equals $\frac{1}{2}$.

(2) Let $\mathcal{H} = M(V)$ with $V = \mathbb{F}^{r|m}$. Observe that V is naturally an irreducible $\mathcal{H}$-module of type $\mathtt{M}$. A similar (and somewhat simpler) calculation as in (1) shows that the Schur element of V (with respect to the usual matrix trace) equals 1.

We denote by $\mathrm{Irr}(\mathcal{H})$ the complete set of non-isomorphic irreducible $\mathcal{H}$-modules. Let χ_V denote the character of an irreducible $\mathcal{H}$-module V, and write

$$\delta(V) = \begin{cases} 0, & \text{if } V \text{ is of type } \mathtt{M}, \\ 1, & \text{if } V \text{ is of type } \mathtt{Q}. \end{cases}$$

Proposition 5.4. *Suppose that $\mathcal{H}$ is a split semisimple superalgebra over $\mathbb{F}$. Then the Schur element c_V for every irreducible $\mathcal{H}$-module V is nonzero. Moreover,*

$$\phi = \sum_{V\in \mathrm{Irr}(\mathcal{H})} \frac{1}{2^{\delta(V)}c_V}\chi_V.$$

Proof. Write $\mathcal{H}$ as a direct sum of simple superalgebras:

$$\mathcal{H} = \bigoplus_{V\in \mathrm{Irr}(\mathcal{H})} H(V). \tag{5.1}$$

Then the irreducible characters χ_V can be identified with the usual matrix trace on $H(V)$. By Remark 5.1, the restriction of the trace form ϕ to $H(V)$ is a scalar multiple of the irreducible character χ_V for each $V \in \mathrm{Irr}(\mathcal{H})$, i.e.,

$$\phi = \sum_{V\in \mathrm{Irr}(\mathcal{H})} d_V\cdot \chi_V$$

for some scalar $d_V \in \mathbb{F}$, which must be nonzero thanks to the non-degeneracy of ϕ. Let $\mathcal{B} = \cup_{V\in \mathrm{Irr}\mathcal{H}}\mathcal{B}(V)$ be a homogeneous basis of $\mathcal{H}$ which is compatible with the decomposition (5.1) and let $\widetilde{\mathcal{B}}(V)$ be the basis in $\mathcal{H}(V)$ dual to $\mathcal{B}(V)$ with respect to the trace function χ_V on $H(V)$. Then $\cup_{V\in \mathrm{Irr}(\mathcal{H})}\{d_V^{-1}b | b\in \widetilde{\mathcal{B}}(V)\}$ is the basis dual to $\mathcal{B}$ with respect to the trace form ϕ. Now fix an irreducible $\mathcal{H}$-module V. For $f\in \mathrm{End}_{\mathbb{F}}(V)_{\bar{0}}$ and $v\in V$, we have

$$\begin{aligned} c_V \mathrm{tr}(f)v =& I(f)(v) = \sum_{b\in\mathcal{B}} b^\vee f(bv)\\ =& \sum_{V'\in \mathrm{Irr}\mathcal{H}}\sum_{b\in \mathcal{B}(V')} b^\vee f(bv) = \sum_{b\in \mathcal{B}(V)} b^\vee f(bv)\\ =& \frac{1}{d_V}\sum_{b\in\mathcal{B}(V)} \widetilde{b} f(bv) = \frac{1}{d_V}\frac{1}{2^{\delta(V)}}\mathrm{tr}(f)(v), \end{aligned}$$

where the fourth equality is due to $bv = 0$ for $b\in \mathcal{B}(V')$ with $V'\neq V$ and the last equality follows from Example 5.3 and the fact that the summation on the right hand side is the defining formula for the Schur element of V with respect to the usual matrix trace χ_V on $H(V)$. Therefore $c_V = \frac{1}{2^{\delta(V)}d_V}$, and the proposition follows. □

Remark 5.5. As in [GP2, Corollary 7.2.4], the following orthogonality relation between split simple characters χ_V and $\chi_{V'}$ holds for a symmetric superalgebra $\mathcal{H}$:

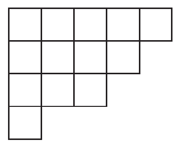
$$\sum_{b\in\mathcal{B}} \chi_V(b)\chi_{V'}(b^\vee) = \begin{cases} 2^{\delta(V)}c_V \mathrm{dim}V, & \text{if } \chi_V = \chi_{V'},\\ 0, & \text{otherwise.}\end{cases}$$

5.2. **The symmetrizing trace form $\daleth$ and Schur elements.** Define a trace function $\daleth : \mathcal{H}^c_{n,\mathbf{A}} \to \mathbf{A}$ which is characterized by the conditions

$$\begin{aligned} \daleth(T_{w_\nu}) &= \Big(\frac{v-1}{2}\Big)^{n-\ell(\nu)}, \qquad \text{for } \nu\in \mathcal{OP}_n,\\ \daleth(z) &= 0, \qquad \text{for } z\in (\mathcal{H}^c_{n,\mathbf{A}})_{\bar{1}}. \end{aligned}$$

By Theorem 4.8 and Corollary 4.9, $\daleth$ is well-defined and unique. We still denote by $\daleth$ the corresponding trace function on $\mathcal{H}^c_{n,\mathbb{K}}$ by a base change. We shall compute the Schur elements for $\mathcal{H}^c_{n,\mathbb{K}}$ with respect to $\daleth$. We first prepare some notations.

Given a partition λ, suppose that the main diagonal of the Young diagram λ contains r cells. Let $\alpha_i = \lambda_i - i$ be the number of cells in the ith row of λ strictly to the right of (i,i), and let $\beta_i = \lambda'_i - i$ be the number of cells in the ith column of λ strictly below (i,i), for $1 \le i \le r$. We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r \ge 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \ge 0$. Then the Frobenius notation for a partition is $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$. For example, if $\lambda = (5,4,3,1)$ whose corresponding Young diagram is

$\lambda =$ 

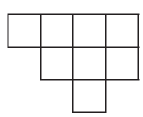
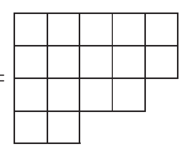
then $\alpha = (4,2,0), \beta = (3,1,0)$ and hence $\lambda = (4,2,0|3,1,0)$ in Frobenius notation.

Suppose that λ is a strict partition of n. Let λ^* be the associated *shifted diagram*, that is,

$$\lambda^* = \{(i,j) \mid 1 \le i \le l(\lambda), i \le j \le \lambda_i + i - 1\}$$

which is obtained from the ordinary Young diagram by shifting the kth row to the right by $k-1$ squares, for each k. Denoting $\ell(\lambda) = \ell$, we define the *double partition* $\widetilde{\lambda}$ to be $\widetilde{\lambda} = (\lambda_1, \dots, \lambda_\ell | \lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_\ell - 1)$ in Frobenius notation. Clearly, the shifted diagram λ^* coincides with the part of $\widetilde{\lambda}$ that lies strictly above the main diagonal. For each cell $(i,j) \in \lambda^*$, denote by h^*_{ij} the associated hook length in the Young diagram $\widetilde{\lambda}$, and set the content $c_{ij} = j - i$.

Example 5.6. *Let $\lambda = (4,3,1)$. The corresponding shifted diagram λ^* and double diagram $\widetilde{\lambda}$ are*

$\lambda^* =$  $\widetilde{\lambda} =$ 

The contents of λ are listed in the corresponding cell of λ^* as follows:

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ & 0 & 1 & 2 \\ & & 0 & \end{array}$$

The shifted hook lengths for each cell in λ^* are the usual hook lengths for the corresponding cell in λ^*, as part of the double diagram $\widetilde{\lambda}$, as follows:

$$\begin{array}{ccccc} \, & 7 & 5 & 4 & 2 \\ \, & \, & 4 & 3 & 1 \\ \, & \, & \, & 1 & \\ \, & \, & & & \end{array} \qquad \begin{array}{cccc} 7 & 5 & 4 & 2 \\ & 4 & 3 & 1 \\ & & 1 & \end{array}$$

For $\lambda \in \mathcal{SP}_n$, let $Q_\lambda(v^\bullet) := Q_\lambda(1, v, v^2, \dots)$ be the principal specialization of Schur Q-function Q_λ at $v^\bullet = (1, v, v^2, \dots)$. The following formula for $Q_\lambda(v^\bullet)$ appeared as [WW1, Theorem B] (also see [Ro] for a different form).

Proposition 5.7. *Suppose $\lambda \in \mathcal{SP}_n$. Then*

$$Q_\lambda(v^\bullet) = \frac{v^{n(\lambda)} \prod_{\square\in\lambda^*}(1+v^{c_\square})}{\prod_{\square\in\lambda^*}(1-v^{h^*_\square})}.$$

Now we compute the Schur elements for simple $\mathcal{H}^c_{n,\mathbb{K}}$-modules.

Theorem 5.8. *$\daleth$ is a symmetrizing trace form on $\mathcal{H}^c_{n,\mathbb{K}}$. For $\lambda \in \mathcal{SP}_n$, the Schur element c^λ of the simple $\mathcal{H}^c_{n,\mathbb{K}}$-module U^λ with respect to $\daleth$ is given by*

$$c^\lambda = 2^{n+\frac{\ell(\lambda)-\delta(\lambda)}{2}} \frac{\prod_{\square\in\lambda^*}(1-v^{h^*_\square})}{v^{n(\lambda)}(1-v)^n \prod_{\square\in\lambda^*}(1+v^{c_\square})}. \tag{5.2}$$

Proof. Set u_λ to be the inverse of the right hand side of (5.2). Recall from Proposition 2.5 that $\mathcal{H}^c_{n,\mathbb{K}}$ is semisimple. By Proposition 5.4, in order to establish the theorem, it suffice to show that

$$\daleth = \sum_{\lambda\in\mathcal{SP}_n} u_\lambda \zeta^\lambda. \tag{5.3}$$

Recall the function $\widetilde{g}_r(x;v)$ from (3.5). Specializing (3.4) at $x = v^\bullet$, we obtain that

$$\sum_{n\geq 0} \tilde{g}_n(v^\bullet;v)t^n = \frac{1}{v-1}\cdot\frac{1-t}{1+t} = \frac{1}{v-1}\big(1+\sum_{n\geq 1}2(-1)^n t^n\big).$$

Hence we have

$$\tilde{g}_n(v^\bullet;v) = \frac{2(-1)^n}{v-1}, \qquad n\geq 1,$$

and

$$\tilde{g}_\mu(v^\bullet;v) = \frac{2^{\ell(\mu)}(-1)^n}{(v-1)^{\ell(\mu)}}, \text{ for } \mu\in\mathcal{P}_n. \tag{5.4}$$

By the Frobenius formula in Theorem 3.7 and the definition of $\daleth$, we obtain that

$$\sum_{\lambda\in\mathcal{SP}_n} 2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} Q_\lambda(v^\bullet)\zeta^\lambda(T_{w_\nu}) = \frac{2^{\ell(\nu)}(-1)^n}{(v-1)^{\ell(\nu)}} = \frac{2^n}{(1-v)^n}\daleth(T_{w_\nu}) \tag{5.5}$$

for all $\nu \in \mathcal{OP}_n$. Then by Corollary 4.9, one deduces that

$$\daleth = \sum_{\lambda\in\mathcal{SP}_n} 2^{-n-\frac{\ell(\lambda)+\delta(\lambda)}{2}}(1-v)^n Q_\lambda(v^\bullet)\zeta^\lambda.$$

Now (5.3) follows from this identity and Proposition 5.7. The theorem is proved. $\square$

It follows from the definition of $\daleth$ that $\daleth(T_{w_\nu}) = \left(\frac{v-1}{2}\right)^{n-\ell(\nu)}$ for odd partition ν of n. The following states that the formula actually hold for all partitions of n.

Corollary 5.9. *For all $\mu\in\mathcal{P}_n$, we have:*

$$\daleth(T_{w_\mu}) = \left(\frac{v-1}{2}\right)^{n-\ell(\mu)}.$$

Proof. Let $\mu \in \mathcal{P}_n$. By Theorem 5.8 (or equivalently, (5.3)), we obtain

$$\begin{aligned}
\daleth(T_{w_\mu}) &= \sum_{\lambda\in\mathcal{SP}_n} 2^{-n-\frac{\ell(\lambda)+\delta(\lambda)}{2}}(1-v)^n Q_\lambda(v^\bullet)\zeta^\lambda(T_{w_\mu}) \\
&= (\frac{1-v}{2})^n \sum_{\lambda\in\mathcal{SP}_n} 2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} Q_\lambda(v^\bullet)\zeta^\lambda(T_{w_\mu}) \\
&= (\frac{1-v}{2})^n \tilde{g}_\mu(v^\bullet; v) \\
&= \Big(\frac{v-1}{2}\Big)^{n-\ell(\mu)},
\end{aligned}$$

where the last two equalities are due to Theorem 3.7 and (5.4), respectively. This proves the corollary. □

5.3. **The generic degrees for $\mathcal{H}^c_{n,\mathbb{K}}$.** Denote by $P_n = \sum_{\sigma\in S_n} v^{\ell(\sigma)}$ the Poincaré polynomial of the symmetric group S_n, and we can formally regard $2^n P_n$ as the Poincaré polynomial of $\mathcal{H}^c_n$. It is known that the Poincaré polynomial P_n is given by

$$P_n = \frac{(1-v)(1-v^2)\cdots(1-v^n)}{(1-v)^n}.$$

Define the *spin generic degree* $D^\lambda = D^\lambda(v)$ associated to the irreducible $\mathcal{H}^c_{n,\mathbb{K}}$-module U^λ, for $\lambda \in \mathcal{SP}_n$, to be

$$D^\lambda = \frac{2^n P_n}{c^\lambda}.$$

The following is a reformulation of Theorem 5.8 by definition of spin generic degrees.

Theorem 5.10. *The following formula for the spin generic degrees holds: for $\lambda \in \mathcal{SP}_n$,*

$$D^\lambda = 2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}} \frac{v^{n(\lambda)}(1-v)1-v^2)\cdots(1-v^n)\prod_{\square\in\lambda^*}(1+v^{c_\square})}{\prod_{\square\in\lambda^*}(1-v^{h^*_\square})}.$$

Remark 5.11. Note that the specialization $\daleth$ at $v=1$ recovers the standard symmetrizing trace form on $\mathfrak{H}^c_n$, which is a twisted group algebra of a double cover of the hyperoctahedral group. Moreover, the specialization

$$D^\lambda|_{v=1} = 2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}} \frac{n!}{\prod_{\square\in\lambda^*} h^*_\square}$$

is the degree of the irreducible $\mathcal{H}^c_n$-module U^λ. Our definition of spin generic degrees for Hecke-Clifford algebras is analogous to Hecke algebras $\mathcal{H}_W$ associate to finite Weyl groups W (cf. [GP2, Section 8.1.8]). The canonical symmetrizing trace form τ on $\mathcal{H}_W$ satisfies $\tau(1)=1$ and $\tau(T_\sigma)=0$ for $1\neq\sigma\in W$.

Remark 5.12. Though various connections between characters and generic degrees of Hecke algebras have been explored in literature, our approach of deriving the closed formula for D^λ directly from the Frobenius character formula is quite elegant and seems to be new even in the usual Hecke algebra setting. We hope to apply the same strategy elsewhere to revisit the generic degrees (or more general notion of weights) for Hecke algebras.

5.4. **Spin fake degrees for the symmetric group.** In this subsection, we shall take $\mathbb{F} = \mathbb{C}$. The symmetric group S_n acts on $\mathbb{C}^n$ and then on the symmetric algebra $S^*\mathbb{C}^n$, which is identified with $\mathbb{C}[x_1, \ldots, x_n]$ naturally. It is well known that the algebra of S_n-invariant on $S^*\mathbb{C}^n$ is a polynomial algebra in the elementary symmetric polynomials $e_1, \ldots, e_n$. The coinvariant algebra of S_n is defined to be

$$(S^*\mathbb{C}^n)_{S_n} = S^*\mathbb{C}^n / I,$$

where I denotes the ideal generated by $e_1, \ldots, e_n$. By a classical theorem of Chevalley the coinvariant algebra $(S^*\mathbb{C}^n)_{S_n}$ is a graded regular representation of S_n. Following Lusztig [Lu], the graded multiplicity of the Specht modules S^λ of S_n in the coinvariant algebra is known as the fake degree of S^λ, for $\lambda \in \mathcal{P}_n$ (cf. [GP2, Section 5.3.3]).

Note that the induced module $\operatorname{ind}_{\mathbb{C}S_n}^{\mathfrak{H}_n^c}(S^*\mathbb{C}^n)_{S_n}$ is a graded regular representation of $\mathfrak{H}_n^c$. Recall from [WW3] that the *spin fake degree* of the irreducible $\mathfrak{H}_n^c$-module U_1^λ with $\lambda \in \mathcal{SP}_n$ is defined to be

$$d^\lambda(t) = \sum_{j \geq 0} t^j \dim \operatorname{Hom}_{\mathfrak{H}_n^c}\big(U_1^\lambda, \operatorname{ind}_{\mathbb{C}S_n}^{\mathfrak{H}_n^c}(S^j\mathbb{C}^n)_{S_n}\big).$$

The spin fake degrees have been computed in [WW1, Theorem A] (though the terminology was introduced later; see [WW3, Theorem 5.8]). A comparison with Theorem 5.10 leads to the following.

Corollary 5.13. *The spin generic degrees coincides with the spin fake degrees, that is,*

$$D^\lambda(v) = d^\lambda(v), \qquad \text{for all } \lambda \in \mathcal{SP}_n.$$

This is parallel to the classical fact (due to Steinberg [S], cf. [Lu, GP2]) that the generic degrees for the Hecke algebra $\mathcal{H}_n$ associated to the symmetric group S_n coincide with the fake degrees for S_n, which is a type A phenomenon.

6. Trace functions on the spin Hecke algebra

6.1. **The spin Hecke algebra $\mathcal{H}_n^-$.** Recall [W] that the *spin Hecke algebra* $\mathcal{H}_n^-$ is a $\mathbb{C}(v^{\frac{1}{2}})$-superalgebra generated by the odd elements $R_i, 1 \leq i \leq n-1$, subject to the following relations:

$$R_i^2 = -(v^2 + 1) \tag{6.1}$$

$$R_i R_j = -R_j R_i \quad (|i - j| > 1) \tag{6.2}$$

$$R_i R_{i+1} R_i - R_{i+1} R_i R_{i+1} = (v-1)^2 (R_{i+1} - R_i). \tag{6.3}$$

Set

$$\begin{aligned} T_i^\Phi &:= -\frac{1}{2} R_i (c_i - c_{i+1}) + \frac{v-1}{2}(1 - c_i c_{i+1}) \in \mathcal{H}_n^- \otimes \mathcal{C}_n, \\ R_i^\Psi &:= (c_i - c_{i+1}) T_i + (v-1) c_{i+1} \in \mathcal{H}_n^c. \end{aligned} \tag{6.4}$$

The tensor superalgebra $\mathcal{H}_n^- \otimes \mathcal{C}_n$ here is understood in the sense of (2.1).

Proposition 6.1. [W] *There exist isomorphisms Φ and Ψ inverse to each other:*

$$\Phi : \mathcal{H}^c_n \longrightarrow \mathcal{H}^-_n \otimes \mathcal{C}_n, \quad \Psi : \mathcal{H}^-_n \otimes \mathcal{C}_n \longrightarrow \mathcal{H}^c_n$$
$$\Phi(T_i) = T_i^{\Phi}, \quad \Phi(c_i) = c_i,$$
$$\Psi(R_i) = R_i^{\Psi}, \quad \Psi(c_i) = c_i, \quad \textit{for all admissible } i.$$

Set $\mathcal{H}^-_{n,\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{C}(v^{\frac{1}{2}})} \mathcal{H}^-_n$. It is known that the Clifford algebra $\mathcal{C}_n$ is a simple superalgebra with a unique irreducible module U_n, which is of type $\mathtt{M}$ if n is even and of type $\mathtt{Q}$ if n is odd. Moreover

$$\dim U_n = \begin{cases} 2^k, & \text{if } n = 2k, \\ 2^{k+1}, & \text{if } n = 2k+1. \end{cases}$$

Thanks to Lemma 2.1, Proposition 2.5 and the above algebra isomorphisms, one sees that for each $\lambda \in \mathcal{SP}_n$ there exists an irreducible $\mathcal{H}^-_{n,\mathbb{K}}$-module U^λ_- with character ζ^λ_- such that $\{U^\lambda_- \mid \lambda \in \mathcal{SP}_n\}$ is a complete set of non-isomorphic irreducible $\mathcal{H}^-_{n,\mathbb{K}}$-modules, and moreover,

$$U^\lambda \cong \begin{cases} 2^{-1} U^\lambda_- \otimes U_n, & \text{if } n \text{ is odd and } \ell(\lambda) \text{ is even}, \\ U^\lambda_- \otimes U_n, & \text{otherwise}. \end{cases} \tag{6.5}$$

6.2. **The space $(\mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}])_{\bar{0}}$.** We shall convert the study of the trace functions on the Hecke-Clifford algebra $\mathcal{H}^c_n$ in Section 4 to the spin Hecke algebra $\mathcal{H}^-_n$. Recall $\mathbf{A} = \mathbb{Z}[\frac{1}{2}][v, v^{-1}]$. Denote by $\mathcal{H}^-_{n,\mathbf{A}}$ the $\mathbf{A}$-subalgebra of the spin Hecke algebra $\mathcal{H}^-_n$ generated by $R_1, \ldots, R_{n-1}$. For $\sigma \in S_n$ with a fixed arbitrary reduced expression $\underline{\sigma} = s_{i_1} s_{i_2} \ldots$, we denote by $R_{\underline{\sigma}} = R_{i_1} R_{i_2} \cdots$. Then it follows from [W] that $\mathcal{H}^-_{n,\mathbf{A}}$ is a free $\mathbf{A}$-module of rank $n!$ and that $\{R_{\underline{\sigma}} \mid \sigma \in S_n\}$ is an $\mathbf{A}$-basis of $\mathcal{H}^-_{n,\mathbf{A}}$. Hence, the analogues of the isomorphisms Φ and Ψ in Proposition 6.1 (which will be denoted by the same notations) make sense over $\mathbb{K}$ or over $\mathbf{A}$.

Lemma 6.2. *Let $\underline{\sigma}$ be an arbitrary reduced expression of $\sigma \in S_n$, and let $I \subseteq [n]$ be nonempty. Assume that the element $R_{\underline{\sigma}} C_I$ is even. Then $R_{\underline{\sigma}} C_I$ belongs to the commutator subspace $[\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n, \mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n]_{\bar{0}}$.*

Proof. Denote $I = \{i_1, \ldots, i_k\}$. Since $R_{\underline{\sigma}} C_I$ is an even element, we have either (i) $R_{\underline{\sigma}}$ is even and k is even, or (ii) $R_{\underline{\sigma}}$ is odd and k is odd. In both cases, we have

$$\begin{aligned} R_{\underline{\sigma}} C_I &= -c_{i_k} R_{\underline{\sigma}} c_{i_1} \cdots c_{i_{k-1}} \\ &\equiv -(R_{\underline{\sigma}} c_{i_1} \cdots c_{i_{k-1}}) c_{i_k} \mod [\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n, \mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n] \\ &= -R_{\underline{\sigma}} C_I. \end{aligned}$$

Since 2 is invertible in $\mathbf{A}$, the lemma is proved. □

Lemma 6.3. *The space $(\mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}])_{\bar{0}}$ is a free $\mathbf{A}$-module.*

Proof. By Lemma 6.2, the space $(\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n/[\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n, \mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n])_{\bar{0}}$ is spanned by images of the elements $R_{\underline{\sigma}}$ with $\ell(\sigma)$ being even under the projection $\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n \to \mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n, \mathcal{H}^-_{n,\mathbf{A}} \otimes \mathcal{C}_n]$. So the natural map ι from $(\mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}])_{\bar{0}}$ to

$(\mathcal{H}^-_{n,\mathbf{A}}\otimes\mathcal{C}_n/[\mathcal{H}^-_{n,\mathbf{A}}\otimes\mathcal{C}_n,\mathcal{H}^-_{n,\mathbf{A}}\otimes\mathcal{C}_n])_{\bar{0}}$ (which is naturally identified via the isomorphism Φ with $(\mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}},\mathcal{H}^c_{n,\mathbf{A}}])_{\bar{0}}$) is surjective. By a base change, ι extends to a surjective map over $\mathbb{K}$. On the other hand, since $\mathcal{H}^c_{n,\mathbb{K}}$ and $\mathcal{H}^-_{n,\mathbb{K}}$ are split semisimple with simple modules parametrized by $\mathcal{SP}_n$, we have

$$\dim_{\mathbb{K}}(\mathcal{H}^-_{n,\mathbb{K}}/[\mathcal{H}^-_{n,\mathbb{K}},\mathcal{H}^-_{n,\mathbb{K}}])_{\bar{0}}=|\mathcal{SP}_n|=\dim_{\mathbb{K}}(\mathcal{H}^c_{n,\mathbb{K}}/[\mathcal{H}^c_{n,\mathbb{K}},\mathcal{H}^c_{n,\mathbb{K}}])_{\bar{0}}. \tag{6.6}$$

So the map ι is actually an isomorphism over $\mathbb{K}$ and hence over $\mathbf{A}$. Now the lemma follows from the $\mathbf{A}$-freeness of $(\mathcal{H}^c_{n,\mathbf{A}}/[\mathcal{H}^c_{n,\mathbf{A}},\mathcal{H}^c_{n,\mathbf{A}}])_{\bar{0}}$ by Theorem 4.8. □

For a composition $\gamma\in\mathcal{CP}_n$ with $\ell(\gamma)=\ell$, the permutation w_γ (see (3.3)) has a unique reduced expression given by

$$\underline{w_\gamma}=(s_1s_2\ldots s_{\gamma_1-1})(s_{\gamma_1+1}\ldots s_{\gamma_1+\gamma_2-1})\cdots(s_{\gamma_1+\ldots+\gamma_{\ell-1}+1}\ldots s_{n-1}).$$

Lemma 6.4. *Let γ be a composition of n with $\ell(w_\gamma)$ being even and μ be the corresponding partition of γ. The following holds:*

(1) *If $\mu\notin\mathcal{OP}_n$, then $R_{\underline{w_\gamma}}\equiv 0 \mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}$.*
(2) *If $\mu\in\mathcal{OP}_n$, then $R_{\underline{w_\gamma}}\equiv R_{\underline{w_\mu}} \mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}$.*

Proof. Suppose $\mu\notin\mathcal{OP}_n$ and $\gamma=(\gamma_1,\ldots,\gamma_\ell)$. Let a be the smallest integer such that γ_a is even, and let b be the smallest integer such that $b>a$ and γ_b is even (which exists as $\ell(w_\gamma)$ is even). Write

$$R_{\underline{w_\gamma}}=R_{\gamma,1}R_{\gamma,2}\cdots R_{\gamma,\ell},$$

where $R_{\gamma,k}=R_{\gamma_1+\ldots+\gamma_{k-1}+1}\cdots R_{\gamma_1+\ldots+\gamma_{k-1}+\gamma_k-1}$ for $1\le k\le\ell$. Then

$$\begin{aligned}R_{\underline{w_\gamma}}\equiv&(R_{\gamma,a}R_{\gamma,a+1}\cdots R_{\gamma,b-1}R_{\gamma,b}R_{\gamma,b+1}\cdots R_{\gamma,\ell})R_{\gamma,1}R_{\gamma,2}\cdots R_{\gamma,a-1}\\ =&R_{\gamma,a}R_{\gamma,b}R_{\gamma,a+1}\cdots R_{\gamma,b-1}R_{\gamma,b+1}\cdots R_{\gamma,\ell}R_{\gamma,1}R_{\gamma,2}\cdots R_{\gamma,a-1}\\ =&-R_{\gamma,b}R_{\gamma,a}R_{\gamma,a+1}\cdots R_{\gamma,b-1}R_{\gamma,b+1}\cdots R_{\gamma,\ell}R_{\gamma,1}R_{\gamma,2}\cdots R_{\gamma,a-1}\\ \equiv&-R_{\gamma,a}R_{\gamma,a+1}\cdots R_{\gamma,b-1}R_{\gamma,b+1}\cdots R_{\gamma,\ell}R_{\gamma,1}R_{\gamma,2}\cdots R_{\gamma,a-1}R_{\gamma,b}\\ =&-R_{\underline{w_\gamma}},\end{aligned}$$

where $\equiv$ is understood $\mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}$ here and below. Therefore, $R_{\underline{w_\gamma}}\equiv 0$.

Now suppose $\mu\in\mathcal{OP}_n$. Using an argument similar to Lemma 3.6, one can obtain that $\zeta^\lambda(R^\Psi_{\underline{w_\gamma}})=\zeta^\lambda(R^\Psi_{\underline{w_\mu}})$ for every irreducible character ζ^λ of $\mathcal{H}^c_{n,\mathbb{K}}$. This implies that

$$\zeta^\lambda_-(R_{\underline{w_\gamma}})=\zeta^\lambda_-(R_{\underline{w_\mu}}),\qquad\text{for each }\lambda\in\mathcal{SP}_n.$$

This together with Lemma 6.3 implies that $R_{\underline{w_\gamma}}\equiv R_{\underline{w_\mu}} \mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}$. □

Lemma 6.5. *Suppose that w_C is a minimal length representative in the conjugacy class C of cycle type $\mu\in\mathcal{P}_n$. Then,*

$$R_{\underline{w_C}}\equiv\begin{cases}\pm R_{\underline{w_\mu}} \mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}], & \text{if } \mu\in\mathcal{OP}_n,\\ 0 \mod [\mathcal{H}^-_{n,\mathbf{A}},\mathcal{H}^-_{n,\mathbf{A}}], & \text{otherwise.}\end{cases}$$

Proof. Suppose $\mu = (\mu_1, \dots, \mu_\ell)$ with $\ell(\mu) = \ell$. The minimal element w_C must be of the form

$$w_C = (s_{i_1^1} s_{i_2^1} \cdots s_{i^1_{\gamma_1-1}})(s_{i_1^2} s_{i_2^2} \cdots s_{i^2_{\gamma_2-1}}) \cdots (s_{i_1^\ell} s_{i_2^\ell} \cdots s_{i^\ell_{\gamma_\ell-1}}),$$

where $\gamma = (\gamma_1, \dots, \gamma_\ell)$ is a composition obtained by rearranging the parts of μ and

$$\{i_1^k, i_2^k, \dots, i^k_{\gamma_k-1}\} = \{\gamma_1 + \cdots + \gamma_{k-1} + 1, \dots, \gamma_1 + \cdots + \gamma_{k-1} + \gamma_k - 1\}$$

for $1 \le k \le \ell$. Recall from (3.3) that w_γ is the permutation associated to γ.

Claim. We have $R_{\underline{w_C}} \equiv \pm R_{\underline{w_\gamma}} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}$.

Indeed, one reduces quickly the proof of the claim to the case $\gamma = (n)$. In this case, we write $w_C = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}$ with $i_a \neq i_b$ for $1 \le a \neq b \le n-1$. If $i_j = j$ for $1 \le j \le n-1$, then $w_C = w_\gamma$. Otherwise, suppose a is the smallest integer such that $i_a \neq a$. We shall prove the claim for $\gamma = (n)$ by reverse induction on a. Observe that $i_a > a$, and hence

$$\begin{aligned} R_{\underline{w_C}} &= R_1 R_2 \cdots R_{a-1} R_{i_a} R_{i_{a+1}} \cdots R_{i_{n-1}} \\ &= (-1)^{a-1} R_{i_a} R_1 R_2 \cdots R_{a-1} R_{i_{a+1}} \cdots R_{i_{n-1}} \\ &\equiv (-1)^{a-1} R_1 R_2 \cdots R_{a-1} R_{i_{a+1}} \cdots R_{i_{n-1}} R_{i_a} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]. \end{aligned}$$

If $i_{a+1} \neq a$ we can apply the above argument again to $R_1 R_2 \cdots R_{a-1} R_{i_{a+1}} \cdots R_{i_{n-1}} R_{i_a}$ to move $R_{i_{a+1}}$ to the end. By repeating the procedure, we obtain that

$$R_{\underline{w_C}} \equiv \pm R_{\underline{w'_C}} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}],$$

where w'_C is a reduced expression of the form $w'_C = s_1 s_2 \cdots s_a s_{i'_{a+1}} \cdots s_{i'_{n-1}}$. Then by induction assumption, we have

$$R_{\underline{w'_C}} \equiv \pm R_{\underline{w_\gamma}} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}].$$

Therefore the claim is proved. Now the lemma follows from Lemma 6.4. ∎

Theorem 6.6. *Let $\sigma \in S_n$ with $\ell(\sigma)$ even and let $\underline{\sigma}$ be a reduced expression of σ. Then there exist $f^-_{\underline{\sigma},\nu} \in \mathbf{A}$ such that*

$$R_{\underline{\sigma}} \equiv \sum_{\nu \in \mathcal{OP}_n} f^-_{\underline{\sigma},\nu} R_{\underline{w_\nu}} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}.$$

Proof. It is more flexible to use induction to establish the following.

Claim. For $\sigma \in S_n$ with $\ell(\sigma)$ being even and an arbitrary reduced expression $r(\sigma)$, there exist constants $f^-_{r(\sigma),\gamma} \in \mathbf{A}$ with $\gamma \in \mathcal{CP}_n$ such that

$$R_{r(\sigma)} = \sum_{\gamma \in \mathcal{CP}_n, \ell(w_\gamma) \text{ even}} f^-_{r(\sigma),\gamma} R_{\underline{w_\gamma}} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]_{\bar{0}}.$$

Note that the theorem follows immediately by the claim, Lemma 6.4 and Lemma 6.5.

To prove the claim, we will follow an approach similar to the proof of Lemma 4.1 or [Ram, Theorem 5.1]. Let i be the smallest integer such that $\sigma(i) > i + 1$. We shall use the induction on $\ell(\sigma)$ and $\sigma(i)$, and reverse induction on i. Note that if there does not exist such i, this can be regarded as the case $i = n$ and σ much be of the form w_γ for

some $\gamma \in \mathcal{CP}_n$. Thus, σ has the unique reduced expression $r(\sigma) = \underline{w_\gamma}$ and the claim follows.

Let $j = \sigma(i) - 1$. Since $\sigma(\sigma^{-1}(j)) = j = \sigma(i) - 1 > i$, the choice of i implies that $\sigma^{-1}(j) > i$. This together with $\sigma^{-1}(j+1) = i$ implies that $\sigma^{-1}(j) > \sigma^{-1}(j+1)$ and hence $\ell(\sigma^{-1}s_j) < \ell(\sigma^{-1})$, or equivalently, $\ell(s_j\sigma) < \ell(\sigma)$. Let $\sigma' = s_j\sigma$ and let $r(\sigma')$ be a reduced expression of σ'. Then $r(\sigma)$ and $s_j r(\sigma')$ are two reduced expressions for σ. By the defining relations among R_i, we have

$$R_{r(\sigma)} = \pm R_j R_{r(\sigma')} + \sum_{\ell(w)<\ell(\sigma)} a_{\sigma,w} R_{r(w)},$$

where $a_{\sigma,w} \in \mathbf{A}$. By induction on $\ell(\sigma)$, we may assume the claim holds for $R_{r(w)}$ for w of length less than σ. Hence we are reduced to show the claim holds for $R_{s_j r(\sigma')} = R_j R_{r(\sigma')}$. Let $\sigma'' = \sigma' s_j$.

If $\ell(\sigma'') > \ell(\sigma')$, then $r(\sigma'') := r(\sigma')s_j$ is a reduced expression of σ'' and hence

$$\begin{aligned} R_j R_{r(\sigma')} &\equiv R_{r(\sigma')} R_j \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}] \\ &= R_{r(\sigma'')} \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]. \end{aligned}$$

Then using the argument similar to the proof of Lemma 4.1, the induction on i and reverse induction on $\sigma(i)$ apply to σ'', and the claim follows.

Otherwise, assume $\ell(\sigma'') < \ell(\sigma')$. Fix a reduced expression $r(\sigma'')$ for σ''. Then $r(\sigma')$ and $r(\sigma'')s_j$ are two reduced expression for σ' and again by defining relations among R_i, we have

$$R_{r(\sigma')} = \pm R_{r(\sigma'')} R_j + \sum_{\ell(w)\le\ell(\sigma'')} b_w R_{r(w)},$$

where $b_w \in \mathbf{A}$. Hence,

$$\begin{aligned} R_j R_{r(\sigma')} &\equiv R_{r(\sigma')} R_j \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}] \\ &= \pm R_{r(\sigma'')} R_j^2 + \sum_{\ell(w)\le\ell(\sigma'')} b_w R_{r(w)} R_j \mod [\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]. \end{aligned}$$

Since $\ell(r(w)s_j) \le \ell(\sigma'') + 1 < \ell(\sigma)$, induction on the length of σ applies to the second summand, and the first summand is also clear since $\ell(\sigma'') < \ell(\sigma)$ and $R_j^2 = -(1+v^2)$.

This completes the proof of the claim and hence the theorem. □

Corollary 6.7. $(\mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}])_{\bar{0}}$ *is an* $\mathbf{A}$*-free module, with a basis consisting of the images of* $R_{\underline{w_\nu}}$ *for* $\nu \in \mathcal{OP}_n$ *under the projection* $\mathcal{H}^-_{n,\mathbf{A}} \to \mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}]$. *Every trace function on* $\mathcal{H}^-_{n,\mathbf{A}}$ *is uniquely determined by its values on* $R_{\underline{w_\nu}}$ *for* $\nu \in \mathcal{OP}_n$.

Proof. The $\mathbf{A}$-freeness follows from Lemma 6.3. By Theorem 6.6, the images of $R_{\underline{w_\nu}}$ (denoted by $\overline{R}_{\underline{w_\nu}}$) for $\nu \in \mathcal{OP}_n$ span $(\mathcal{H}^-_{n,\mathbf{A}}/[\mathcal{H}^-_{n,\mathbf{A}}, \mathcal{H}^-_{n,\mathbf{A}}])_{\bar{0}}$. By passing to the field $\mathbb{K}$ and a dimension counting (6.6), we see that these elements $\overline{R}_{\underline{w_\nu}}$ are linearly independent. The corollary follows. □

The character table for Hecke algebras or Hecke-Clifford algebras has a natural generalization for spin Hecke algebra as follows. The matrix

$$(\zeta_-^\lambda(R_{\underline{w_\nu}}))_{\lambda\in\mathcal{SP}_n,\nu\in\mathcal{OP}_n}$$

is called the *character table* of the spin Hecke algebra $\mathcal{H}^-_{n,\mathbb{K}}$ over $\mathbb{K}$. By Corollary 6.7 and the linear independence of irreducible characters ζ_-^λ for $\lambda \in \mathcal{SP}_n$, the character table $(\zeta_-^\lambda(R_{\underline{w_\nu}}))_{\lambda\in\mathcal{SP}_n,\nu\in\mathcal{OP}_n}$ is invertible.

It follows by Corollary 6.7 that $f^-_{\underline{\sigma},\nu}$ in Theorem 6.6 is uniquely determined by $\underline{\sigma}$ and ν. Similar to [GP1], $f^-_{\underline{\sigma},\nu}$ will be called the *class polynomials* of spin Hecke algebras. By Corollary 6.7, there exists a unique function $f^-_\nu : \mathcal{H}^-_{n,\mathbf{A}} \to \mathbf{A}$ characterized by

$$f^-_\nu(R_{\underline{w_\rho}}) = \delta_{\nu,\rho}, \qquad \text{for } \rho \in \mathcal{OP}_n.$$

By Theorem 6.6, for an arbitrary reduced expression $\underline{\sigma}$ of $\sigma \in S_n$, we have

$$f^-_\nu(R_{\underline{\sigma}}) = \begin{cases} f^-_{\underline{\sigma},\nu}, & \text{for } \ell(\sigma) \text{ even},\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 6.6 and Corollary 6.7 imply the following.

Proposition 6.8. *The functions f^-_ν for $\nu \in \mathcal{OP}_n$ form a basis of the space of trace functions on $\mathcal{H}^-_{n,\mathbf{A}}$.*

6.3. **The trace form $\daleth^-$ on $\mathcal{H}^-_n$.** The trace form $\daleth$ on $\mathcal{H}^c_{n,\mathbb{K}}$ induces a symmetrizing trace form, which will be still denoted by $\daleth$, on $\mathcal{H}^-_{n,\mathbb{K}} \otimes \mathcal{C}_{n,\mathbb{K}}$ via the isomorphism $\mathcal{H}^c_{n,\mathbb{K}} \cong \mathcal{H}^-_{n,\mathbb{K}} \otimes \mathcal{C}_{n,\mathbb{K}}$, where $\mathcal{C}_{n,\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{C}} \mathcal{C}_n$. This in turn restricts to a symmetrizing trace form $\daleth^-$ on $\mathcal{H}^-_{n,\mathbb{K}}$ (identified with $\mathcal{H}^-_{n,\mathbb{K}} \otimes 1$).

Proposition 6.9. *For any composition $\mu \neq (1^n)$ of n, $\daleth^-(R_{\underline{w_\mu}}) = 0$.*

Proof. By Lemma 6.4, it suffices to establish the case when $\mu \in \mathcal{OP}_n$. We shall prove by induction on the dominance order of μ.

Observe that the trace form $\daleth$ on $\mathcal{H}^-_n \otimes \mathcal{C}_n$ satisfies that

$$\daleth(T^\Phi_{w_\mu}) = \Big(\frac{v-1}{2}\Big)^{n-\ell(\mu)}, \tag{6.7}$$

where we have denoted $T^\Phi_{w_\mu} = \Phi(T_{w_\mu})$. Recall (6.4) and that

$$w_\mu = (s_1 s_2 \dots s_{\mu_1-1})(s_{\mu_1+1}\dots s_{\mu_1+\mu_2-1})\cdots(s_{\mu_1+\ldots+\mu_{\ell-1}+1}\dots s_{n-1}).$$

We write

$$T^\Phi_{w_\mu} = X_1 + X_2 + X_3, \tag{6.8}$$

where

$$
\begin{aligned}
X_1 =& \Big(-\frac{1}{2}\Big)^{n-\ell(\mu)} R_1(c_1-c_2)\cdots R_{\mu_1-1}(c_{\mu_1-1}-c_{\mu_1})\cdot \\
& R_{\mu_1+1}(c_{\mu_1+1}-c_{\mu_1+2})\cdots R_{\mu_1+\mu_2-1}(c_{\mu_1+\mu_2-1}-c_{\mu_1+\mu_2})\cdots \cdot \\
& \cdot R_{\mu_1+\cdots\mu_{\ell-1}+1}(c_{\mu_1+\cdots\mu_{\ell-1}+1}-c_{\mu_1+\cdots\mu_{\ell-1}+2})\cdots R_{n-1}(c_{n-1}-c_n), \\
X_2 =& \Big(\frac{v-1}{2}\Big)^{n-\ell(\mu)}(1-c_1c_2)\cdots(1-c_{\mu_1-1}c_{\mu_1})(1-c_{\mu_1+1}c_{\mu_1+2})\cdots \\
& \cdot(1-c_{\mu_1+\mu_2-1}c_{\mu_1+\mu_2})\cdots(1-c_{\mu_1+\ldots+\mu_{\ell-1}+1}c_{\mu_1+\ldots+\mu_{\ell-1}+2})\cdots(1-c_{n-1}c_n), \\
X_3 =& \sum_{I\subseteq[n],\gamma\in\mathcal{CP}_n,(1^n)\neq\overline{\gamma}<\mu} a_{\gamma,I}R_{\underline{w_\gamma}}C_I,
\end{aligned}
$$

with $a_{\gamma,I}\in\mathbf{A}$ and $\overline{\gamma}$ denoting the partition corresponding to the composition γ.

It follows by Lemma 6.2 that $\gimel(R_{\underline{w_\gamma}}C_I)=0$ if $I\neq\emptyset$, and by Lemma 6.4 and induction on μ by dominance order that $\gimel(R_{\underline{w_\gamma}})=\gimel^-(R_{\underline{w_\gamma}})=0$ for $\gamma\in\mathcal{CP}_n$ with $\overline{\gamma}<\mu$. Hence

$$\gimel(X_3)=0. \tag{6.9}$$

Note that $X_2=(\frac{v-1}{2})^{n-\ell(\mu)}+$ a linear combination of C_I with $I\neq\emptyset$. By Lemma 6.2,

$$\gimel(X_2)=(\frac{v-1}{2})^{n-\ell(\mu)}. \tag{6.10}$$

For $\mu\in\mathcal{OP}_n$, we have

$$X_1=(-\frac{1}{2})^{n-\ell(\mu)}R_{\underline{w_\mu}}+\sum_{\emptyset\neq I\subseteq[n]} b_I R_{\underline{w_\mu}}C_I$$

for some scalars b_I. Hence, by Lemma 6.2, we have

$$\gimel(X_1)=(-\frac{1}{2})^{n-\ell(\mu)}\gimel(R_{\underline{w_\mu}}). \tag{6.11}$$

Collecting (6.8), (6.9), (6.10) and (6.11), we obtain that

$$\gimel(T^\Phi_{w_\mu})=\Big(\frac{v-1}{2}\Big)^{n-\ell(\mu)}+(-\frac{1}{2})^{n-\ell(\mu)}\gimel(R_{\underline{w_\mu}}).$$

By a comparison with (6.7) we conclude that $\gimel^-(R_{\underline{w_\mu}})=\gimel(R_{\underline{w_\mu}})=0$. □

By convention, we have $R_{\underline{w_{(1^n)}}}=1$. By Lemma 6.5, Theorem 6.6 and Proposition 6.9, we have established the following.

Theorem 6.10. (1) $\gimel^-(R_{\underline{w_C}})=0$ *for any minimal length representative* w_C *in a non-identity conjugacy class* C *of* S_n *with any reduced expression* $\underline{w_C}$.

(2) $\gimel^-$ *is characterized by the property* $\gimel^-(R_{\underline{w_\nu}})=\delta_{\nu,(1^n)}$ *for* $\nu\in\mathcal{OP}_n$.

Example 6.11. *It is possible that* $\gimel^-(R_{\underline{\sigma}})\neq 0$ *if* σ *is not a minimal length element in its conjugacy class. For example, the permutation* $\sigma=(2,3)(1,4)$ *has a reduced expression* $\underline{\sigma}=s_2s_1s_3s_2s_3s_1$, *and one computes that* $\gimel^-(R_{\underline{\sigma}})=-(v-1)^4(v^2+1)$.

Remark 6.12. Using the symmetrizing trace function $\daleth^-$ on $\mathcal{H}_n^-$, we can determine the Schur elements c_-^λ associated to the irreducible characters ζ_-^λ of $\mathcal{H}_{n,\mathbb{K}}^-$. These Schur elements c_-^λ turn out to be related to the Schur elements c^λ associated to the irreducible character ζ^λ of $\mathcal{H}_{n,\mathbb{K}}^c$ (see Theorem 5.8), via

$$c_-^\lambda = \begin{cases} 2^{-k}c^\lambda, & \text{if } n = 2k, \\ 2^{-k-\delta(\lambda)}c^\lambda, & \text{if } n = 2k+1. \end{cases}$$

This can be deduced by using (6.5) and noting that $\daleth$ can be identified with the tensor product of $\daleth^-$ and the usual matrix trace on $\mathcal{C}_n$.

References

[Fr] F.G. Frobenius, Über die Charactere der symmetrischen Gruppe, Sitzungsber. K. Preuss. Akad. Wisse. Berlin, 516–534(1900). reprinted in: Gessamelte Abhandlungen 3, pp. 148–166. Berlin Heidelberg New York: Springer 1973.

[GP1] M. Geck and G. Pfeiffer, *On the irreducible characters of Hecke algebras*, Adv. Math. **102** (1993), 79–94.

[GP2] M. Geck and G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, London Math. Soc. Monographs, New Series **21**, Oxford University Press, New York 2000.

[HKS] D. Hill, J. Kujawa and J. Sussan, *Degenerate affine Hecke-Clifford algebras and type Q Lie superalgebras*, Math. Z **268** (2011), 1091-1158.

[JN] A. Jones and M. Nazarov, *Affine Sergeev algebra and q-analogues of the Young symmetrizers for projective representations of the symmetric group*, Proc. London Math. Soc. **78** (1999), 481–512.

[Jo] T. Józefiak, *A class of projective representations of hyperoctahedral groups and Schur Q-functions*, Topics in Algebra, Banach Center Publ. **26**, Part 2, PWN-Polish Scientific Publishers, Warsaw (1990), 317–326.

[KW] R.C. King and B. Wybourne, *Representations and traces of the Hecke algebras $H_n(q)$ of type A_{n-1}*, J. Math. Phys. **33** (1992), 4–14.

[Lu] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. of Math Stud. **107**, Princeton University Press, 1984.

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Clarendon Press, Oxford, 1995.

[Ol] G.I. Olshanski, *Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra*, Lett. Math. Phys. **24** (1992), 93–102.

[Ram] A. Ram, *A Frobenius formula for the characters of the Hecke algebras*, Invent. Math. **106** (1991), 461–488.

[Ro] H. Rosengren, *Schur Q-polynomials, multiple hypergeometric series and enumeration of marked shifted tableaux*, J. Combin. Theory Ser. **A 115** (2008), 376–406.

[Se] A. Sergeev, *Tensor algebra of the identity representation as a module over the Lie superalgebras $GL(n,m)$ and $Q(n)$*, Math. USSR Sbornik **51** (1985), 419–427.

[Sch] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, J. Reine Angew. Math. **139** (1911), 155–250.

[S] R. Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*, Trans. Amer. Math. Soc. **71** (1951), 274–282.

[Wan] J. Wan, *Completely splittable representations of affine Hecke-Clifford algebras*, J. Algebraic Combin. **32** (2010), 15–58.

[WW1] J. Wan and W. Wang, *Spin invariant theory for the symmetric group*, J. Pure Appl. Algebra **215** (2011), 1569–1581.

[WW2] J. Wan and W. Wang, *Spin Kostka polynomials*, J. Algebraic Combin. (to appear), DOI:10.1007/s10801-012-0362-4, 2012.

[WW3] J. Wan and W. Wang, *Lectures on spin representation theory of symmetric groups*, Bull. Inst. Math. Acad. Sin. (N.S.) **7** (2012), 91–164.

[W] W. Wang, *Spin Hecke algebras of finite and affine types*, Adv. in Math. **212** (2007), 723–748.

Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, P.R. China.

E-mail address: wjk302@gmail.com

Department of Mathematics, University of Virginia, Charlottesville,VA 22904, USA.

E-mail address: ww9c@virginia.edu